Recitation 2

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1 Quantization in Quantum Mechanics

1.1 First, there was Hamilton

Let us review classical mechanics, in the Hamiltonian formalism:

- States: given by a point in phase space $M = \mathbb{R}^{2d}$ with coordinates (q_j, p_j) for $1 \leq j \leq d$
- Observables: functions on phase space, $F(M)$
- Dynamics: states evolve according to Hamilton's equations, $\dot{q}_j = \frac{\partial H}{\partial p_j}$, $\dot{p}_j = -\frac{\partial H}{\partial q_j}$

We can capture the dynamics by defining the Poisson bracket

$$\{\cdot,\cdot\} : F(M) \times F(M) \to F(M), \qquad (f_1, f_2) \mapsto \{f_1, f_2\} = \frac{\partial f_1}{\partial q_i}\frac{\partial f_2}{\partial p_i} - \frac{\partial f_1}{\partial p_i}\frac{\partial f_2}{\partial q_i}$$

- Mathematically, it gives $F(M)$ a group structure (more precisely, a Lie algebra structure, due to antisymmetry + Jacobi identity)
- The Poisson bracket generates time translations: an observable evolves in time

$$\frac{dA}{dt} = \frac{\partial A}{\partial q_i}\dot{q}_i + \frac{\partial A}{\partial p_i}\dot{p}_i + \frac{\partial A}{\partial t} = \{A, H\} + \frac{\partial A}{\partial t}$$

  where the second term vanishes if A does not explicitly depend on time
- Observe: this looks a lot like the equations of motion for operators in the Heisenberg picture

$$\frac{d\mathcal{O}}{dt} = -i[\mathcal{O}, H] + \frac{\partial \mathcal{O}}{\partial t}$$

1.2 Quantization

Quantization: the procedure of turning a classical theory into a valid quantum theory

- Many quantization schemes in QM: Dirac quantization, path-integral quantization, deformation quantization, geometric quantization, etc. Can generalize each of these to QFT.

Axiom: whenever a group G acts on a classical system, in the corresponding quantum theory, the Hilbert space $\mathcal{H}$ carries a unitary representation of G (have 'operators')

Dirac quantization: promote each classical observable f to an self-adjoint operator $\hat{O}_f$ acting on a Hilbert space $\mathcal{H}$ such that $\hat{O}_{\{f,g\}} = -\frac{i}{\hbar}[\hat{O}_f, \hat{O}_g]$ (replace P.B. with commutators)

- If we view $F(M)$ as a group with operation given by the Poisson bracket, quantization gives a unitary representation on $\mathcal{H}$
- For positions and momenta, we have:

$$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0, \qquad [\hat{q}_i, \hat{p}_j] = i\hbar$$

We can try continue this prescription to more complicated functions of q and p, but this breaks down for polynomials of degree ≥ 3 (Groenewold's theorem). In particular, operator ordering ambiguities generate corrections of higher-order in h. To second order in q and p said ordering ambiguities are not fatal, making quantities like the Hamiltonian well-defined.

Usually, we only care about restricting quantization to linear (sometimes quadratic) polynomials in x, p

- In classical mechanics, linear functions in q and p are closed under taking Poisson brackets, generating the Heisenberg group
- In quantum mechanics, we want operators satisfying (taking $\hbar = 1$)

$$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0, \qquad [\hat{q}_i, \hat{p}_j] = i$$

  Existence and uniqueness: given by the Stone-von Neumann theorem

The Stone-von Neumann theorem:

1. There are no finite dimensional representations of the Heisenberg group
   I.e. $\hat{x}$ and $\hat{p}$ cannot be finite dimensional matrices (proof: take trace)
2. All (irreducible) representations of $\hat{x}$ and $\hat{p}$ are unitarily equivalent, and act on the Hilbert space of square-integrable functions $L^2(\mathbb{R})$

This is the most important theorem in quantum mechanics. In particular (2) proves that the following are all equivalent.

- Position-space representation: $\hat{x} \to x, \quad \hat{p} = -i\frac{\partial}{\partial x}$
- Momentum-space representation: $\hat{x} \to i\frac{\partial}{\partial x}, \quad \hat{p} = p$ related by Fourier (unitary!)
- We have a representation at every t. Time evolution, then, relates representations at different times, $(\hat{x}(t), \hat{p}(t), \mathcal{H}(t))$. By Stone-von Neumann all representations of $\hat{x}$ and $\hat{p}$ are unitarily equivalent, so time-evolution must be unitary. There is an inherent ambiguity: we can either evolve the operators (active), the states (passive), or something in-between, but all these are fundamentally the same. This demonstrates the equivalence of the Heisenberg and Schrödinger pictures of QM.

In practice, to quantize a Lagrangian system we do the following.

1. Identify the generalized coordinates (degrees of freedom) q_i of a classical system
2. Derive the conjugate generalized momenta p_i from the Lagrangian.
3. Promote (q_i, p_i) into operators $(\hat{q}_i, \hat{p}_i)$, which satisfy the canonical commutation relations

$$[\hat{q}_i, \hat{p}_j] = i\hbar\{q_i, p_j\} = i\hbar\delta_{ij}$$

Example: the classical harmonic oscillator, $L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2$ (e.o.m. $\ddot{x} - \omega^2 x = 0$)

- Conjugate momentum: $p = \dot{x}$, Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2$
- We can introduce 'normal coordinates' $a = \frac{1}{\sqrt{2\omega}}(\omega x + ip), \quad a^* = \frac{1}{\sqrt{2\omega}}(\omega x - ip), \quad H = \omega a a^*$
- Quantization: can promote (x, p) to operators, or (a, a^*) to operators. Suppose we do the latter.
  the commutator is given by the Poisson bracket

$$[\hat{a}, \hat{a}^*] = i\hbar\{a, a^*\} = i\hbar\left(\frac{\partial a}{\partial x}\frac{\partial a^*}{\partial p} - \frac{\partial a}{\partial p}\frac{\partial a^*}{\partial x}\right) = i\hbar(-i) = 1$$

- The operators $\hat{a}$, $\hat{a}^\dagger$ naturally act on the Bargmann-Fock space of square-integrable holomorphic (complex differentiable) functions. By Stone-von Neumann, there must be a unitary map between this and $L^2(\mathbb{R})$ in the position representation. Indeed there is, you should think of this map as just a change of basis. This map identifies Bargmann-Fock as the space of coherent states. Mathematically, it's nice to work in the basis of coherent states, because unlike with delta-functions and plane waves, I never encounter any problems with infinities or normalizability.

2 Quantization in QFT

2.1 Classical Field Theory

We review some basic field theory. We work in Minkowski space, with a scalar field ϕ. Everything in this discussion can be extended to vector, tensor, and spinor fields with some effort (sometimes, with a lot of effort), which will be the subject of much of this course.

The starting point is the action. This is the time-integral of the Lagrangian L, or the spacetime-integral of the Lagrangian density $\mathcal{L}$.

$$S[\phi] = \int dt L[\phi] = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$$

What it means for ϕ to be a Lorentz scalar is that under a Lorentz transformation Λ, the field transforms as $\phi \to \phi'$, such that $\phi'(\Lambda x) = \phi(x)$. This is a confusing point, so I'll elaborate. [draw picture]

The principle of least action leads to the Euler-Lagrange equations of motion, which governs the time-evolution of ϕ.

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

Symmetries and Noether's theorem. Ask class if they want to go over an example. Use 2-component scalar field ϕ_a with SO(2) symmetry. Angular momentum.

To prepare for quantization, let us pick a specific model. Namely, we work with a free (zero potential) real scalar field of mass m, with Lagrangian and equation of motion given by

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - m^2\phi^2, \qquad (\partial^2 - m^2)\phi(x) = 0$$

The conjugate momentum to ϕ is $\pi = \partial_t \phi = \dot{\phi}$. The Hamiltonian density is thus

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

In order to determine the degrees of freedom of our system, let us examine the equations of motion, which we write in Fourier (momentum) space $\tilde{\phi}(t, \mathbf{k})$:

$$(-\partial_t^2 + \omega_{\mathbf{k}}^2)\tilde{\phi}(t, \mathbf{k}) = 0$$

Here $\omega_{\mathbf{k}}^2 = \mathbf{k}^2 + m^2$, and $\mathbf{k}$ a free parameter. The equation of motion for our field has now been decoupled into its momentum components. For each $\mathbf{k} \in \mathbb{R}^3$ we have such an equation, which is precisely the equation of motion for a harmonic oscillator of frequency $\omega_{\mathbf{k}}$. The moral is that now we can immediately read off our system's degrees of freedom: for each (continuous) momentum value we get a harmonic oscillator. This is a generic feature of free QFTs.

2.2 Canonical Quantization

We are ready to quantize our theory, and in fact we can immediately quote our results from earlier for a single harmonic oscillator. The mode $\tilde{\phi}(t, \mathbf{k})$ becomes an operator $\hat{\tilde{\phi}}(t, \mathbf{k})$ given by

$$\hat{\tilde{\phi}}(t, \mathbf{k}) = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}(\hat{a}_{\mathbf{k}}(t) + \hat{a}_{\mathbf{k}}^\dagger(t)) = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}(\hat{a}_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} t} + \hat{a}_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}} t})$$

To get back to position space we take a Fourier transform to yield a mode expansion:

$$\hat{\phi}(t,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{k}}}(\hat{a}_\mathbf{k} e^{-i\omega_\mathbf{k} t + i\mathbf{k}\cdot\mathbf{x}} + \hat{a}^\dagger_\mathbf{k} e^{i\omega_\mathbf{k} t - i\mathbf{k}\cdot\mathbf{x}})$$

This is an important equation in QFT. We can also compute the canonical momentum:

$$\hat{\pi}(t,\mathbf{x}) = \partial_t\hat{\phi}(t,\mathbf{x}) = -i\int \frac{d^3\mathbf{k}}{(2\pi)^3}\sqrt{\frac{\omega_\mathbf{k}}{2}}(\hat{a}_\mathbf{k} e^{-i\omega_\mathbf{k} t + i\mathbf{k}\cdot\mathbf{x}} - \hat{a}^\dagger_\mathbf{k} e^{i\omega_\mathbf{k} t - i\mathbf{k}\cdot\mathbf{x}})$$

Finally, quantization. There are two ways we can do this.

1. Demanding the canonical commutation relations for a generalized coordinate and conjugate momenta, in this case the pair ϕ and π:

$$[\hat{\phi}(t,\mathbf{x}),\hat{\phi}(t,\mathbf{x}')] = [\hat{\pi}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{x}')] = 0, \qquad [\hat{\phi}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{x}')] = i\delta^{(3)}(\mathbf{x}-\mathbf{x}')$$

2. Using the above insight where we have a continuum of harmonic oscillators we want to quantize:

$$[\hat{a}_\mathbf{k},\hat{a}_{\mathbf{k}'}] = [\hat{a}^\dagger_\mathbf{k},\hat{a}^\dagger_{\mathbf{k}'}] = 0, \qquad [\hat{a}_\mathbf{k},\hat{a}^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')$$

   The prefactor $(2\pi)^3$ is not important, since it can always be removed by rescaling operators. However, it is necessary so that (1) and (2) are consistent, as one can check using the mode expansions for $\hat{\phi}(x^\mu)$ and $\hat{\pi}(x^\mu)$. In fact, (1) and (2) are equivalent in that imposing each commutator implies the other, serving as a good consistency check of our theory.

By imposing the necessary canonical commutation relations, we have thus quantized a classical field theory, paving the way for quantum field theory.

It remains to discuss the Hilbert space of our theory, and again we draw from our knowledge of the quantum harmonic oscillator. $\mathcal{H}$ is generated by acting creation operators $\hat{a}_\mathbf{k}$ on some vacuum state $|0\rangle$. There is an important difference: in QM creation operators create one unit of energy quanta. In QFT, a creation operator $\hat{a}^\dagger_\mathbf{k}$ creates a bona fide particle of momentum $\mathbf{k}$. The total Hilbert space is thus

$$\mathcal{H} = \bigoplus_{n\geq 0}\mathcal{H}_n, \qquad \mathcal{H}_n = \text{span}\{\hat{a}^\dagger_{\mathbf{k}_1}\cdots\hat{a}^\dagger_{\mathbf{k}_n}|0\rangle\}$$

where $\mathcal{H}_n$ is the n-particle sector, and $|0\rangle$ is the vacuum state (Note that $\mathcal{H}_0 = \{|0\rangle\}$). When we do quantum mechanics without QFT, we are locked into working in a particular sector $\mathcal{H}_n$. On the other hand, QFT provides a natural way to move between sectors: particle creation and annihilation are intrinsic to the theory.

A last note. In quantum mechanics, we have position operators $\hat{x}$, but no time operator t. In quantum mechanics time serves a privileged role as a parameter. Special relativity tells us that space and time are intimately related, and so in QFT we demote position to the same status as time, from an operator to just a parameter. Indeed, fields take values in both position and time. There is another way to approach particle physics. Namely, we could have also promoted time to an operator and retain position operators so they are again on equal footing. Doing so, one encounters several problems. For instance, owing to the -1 in the Minkowski metric, the Hilbert space no longer has a positive definite inner product. This is very bad, but not fatal–the theory can be remedied by restricting to a 'physical' Hilbert space. One can proceed to get a consistent quantum theory, but this is hard, less intuitive, and not particularly insightful for us. However, the approach of promoting time to an operator is how string theory is formulated, at least perturbatively.

2.3 The Role of Time in Quantization

I just told you that time and space should be treated equally, since they are part of the same entity we call spacetime. I will proceed to tell you why the opposite is true: even in QFT, time has a very privileged role to play.
Let us return to our scalar equation of motion,

$$(-\partial_t^2 + \omega_{\mathbf{k}}^2)\tilde{\phi}(t, \mathbf{k}) = 0$$

This is a second order ODE, so for a given $\omega_{\mathbf{k}}$ this has 2 solutions: $\tilde{\phi}(t, \mathbf{k}) \propto e^{\pm i\omega_{\mathbf{k}} t}$. One has positive frequency, and the other negative frequencies: it is this which demarcates particles from antiparticles. To be more precise, it is this ∂_t operator whose eigenvalues separates modes into positive and negative frequency. In the language of general relativity, this separation depends on the existence of a global timelike Killing vector, which is a coordinate transformation which preserves the metric. In special relativity we work in Minkowski space, where ∂_t fulfills this role.

More generally, we can quantize a field theory in curved spacetimes that lack a timelike Killing vector. There is thus no natural decomposition of ϕ into positive and negative frequencies, into particles and antiparticles, into creation and annihilation operators. In fact, the notion of a particle in curved space does not have universal significance, and is observer-dependent. A state where one observer observes no particles, another observer might see them. Because the creation and annihilation operators are crucial to defining the vacuum of the theory, different observers will have different vacua. Generically, such vacua are no longer devoid of particles, even if an observer is free-falling. In Minkowski space we don't have this problem because the vacuum state we defined earlier today is the agreed vacuum for all inertial observers, as both are invariant under the Poincaré group.

Let me end off today with 2 experimental consequences of formulating QFT in curved space.

1. Consider 2 observers in Minkowski space, with a scalar field. The first is inertial, and the second is uniformly accelerating at a. Suppose the field is in the vacuum state for the inertial observer, which is just the $|0\rangle$ state we wrote above. Then, the accelerating observer will see a thermal bath of particles, at temperature $T \propto a$. This phenomenon is called Unruh radiation, with T the Unruh temperature.

2. Consider a scalar field in a universe that is asymptotically Minkowski at early and late times, but undergoes a period of inflation from t_1 to t_2. Then, even if the field ϕ is prepared in the vacuum state at time $t = -\infty$, an inertial observer at late times will see particles: this is the phenomenon of particle creation is cosmological spacetimes.

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