# Quantum Field Theory I (8.323) Spring 2023
# Assignment 2

Feb. 14, 2023

- Please remember to put **your name** at the top of your paper.

**Readings**

- Peskin & Schroeder Chap. 2
- Weinberg vol 1 Chap. 1

**Notes:**

1. **Conventions on Fourier transform and the Dirac delta function**

   - Fourier transform of $\phi(\vec{x}, t)$ is defined as

   $$\tilde{\phi}(\vec{k}, \omega) = \int dt d^3\vec{x}\, e^{i\omega t - i\vec{k}\cdot\vec{x}}\, \phi(\vec{x}, t) \tag{1}$$

   with the inverse transform given by

   $$\phi(\vec{x}, t) = \int \frac{d\omega}{2\pi} \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\omega t + i\vec{k}\cdot\vec{x}}\, \tilde{\phi}(\vec{k}, \omega)\ . \tag{2}$$

   We will often suppress the tilde on $\tilde{\phi}(\vec{k}, \omega)$ and simply write it as $\phi(\vec{k}, \omega)$, distinguishing it from $\phi(\vec{x}, t)$ by their arguments.

   - Note

   $$\int_{-\infty}^{\infty} dx\, e^{ikx} = 2\pi\delta(k), \tag{3}$$

   and it higher dimensional generalizations

   $$\int d^3\vec{x}\, e^{i\vec{k}\cdot\vec{x}} = (2\pi)^3\delta^{(3)}(\vec{k}) \tag{4}$$

2. **Lorentz transformations**

- A Lorentz transformation acts on $x^\mu$ and $p^\mu$ as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu, \qquad p^\mu \to p'^\mu = \Lambda^\mu{}_\nu p^\nu \tag{5}$$

where the matrix $\Lambda^\mu{}_\nu$ satisfies the relation

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\lambda \eta^{\rho\lambda} = \eta^{\mu\nu} \tag{6}$$

or in a matrix notation

$$\Lambda \eta \Lambda^t = \eta \tag{7}$$

where the superscript $t$ denotes transpose. We can raise and lower the indices of $\Lambda$ by $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, and equation (7) can also be written as

$$\Lambda_\mu{}^\rho \Lambda_\nu{}^\lambda \eta_{\rho\lambda} = \eta_{\mu\nu} \ . \tag{8}$$

- Under a Lorentz transformation (5), a scalar field transforms as

$$\phi(x) \to \phi'(x') = \phi(x) \ ; \tag{9}$$

a vector field transforms as

$$A_\mu(x) \to A'_\mu(x') = \Lambda_\mu{}^\nu A_\nu(x) \ ; \tag{10}$$

a second rank tensor field transforms as

$$T_{\mu\nu}(x) \to T'_{\mu\nu}(x') = \Lambda_\mu{}^\lambda \Lambda_\nu{}^\rho T_{\lambda\rho}(x) \tag{11}$$

and so on.

- Infinitesimal Lorentz transformations take the form

$$\Lambda_\mu{}^\nu = \delta_\mu{}^\nu + \omega_\mu{}^\nu \tag{12}$$

where

$$\omega_{\mu\nu} = -\omega_{\nu\mu}, \qquad \omega_\mu{}^\nu = \eta^{\nu\lambda}\omega_{\mu\lambda} \tag{13}$$

are infinitesimal numbers.

3. All single-particle states used below follow relativistic normalization, i.e.

$$\boxed{|k\rangle = \sqrt{2\omega_{\vec{k}}}\, a^\dagger_{\vec{k}} |0\rangle} \ . \tag{14}$$

**Problem Set 2**

1. **Problem with relativistic quantum mechanics (20 points)**

   The Schrodinger equation for a free non-relativistic particle is

   $$i\partial_t\psi(\vec{x},t) = -\frac{1}{2m}\nabla^2\psi(\vec{x},t)\ . \tag{15}$$

   The generalization of the above equation to a free relativistic particle is the so-called Klein-Gordon equation

   $$\partial_t^2\psi(\vec{x},t) - \nabla^2\psi(\vec{x},t) + m^2\psi(\vec{x},t) = 0\ . \tag{16}$$

   We emphasize that in both (15) and (16), $\psi(\vec{x},t)$ is interpreted as a wave function for dynamical variable $\vec{x}(t)$ rather than a dynamical field.

   (a) As a reminder, derive from (15) the continuity equation for the probability

   $$\partial_t\rho + \nabla\cdot\vec{J} = 0, \tag{17}$$

   where

   $$\rho = |\psi|^2, \qquad \vec{J} = -\frac{i}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)\ . \tag{18}$$

   (b) Suppose $\psi(\vec{x},t)$ has the plane wave form, i.e.

   $$\psi(\vec{x},t) \propto e^{i\vec{k}\cdot\vec{x}} \tag{19}$$

   for some real vector $\vec{k}$, find the solutions to (16).

   (c) Show that the Klein-Gordon equation also leads to a continuity equation (17) with now $\rho$ and $\vec{J}$ given by

   $$\rho = \frac{i}{2m}\left(\psi^*\partial_t\psi - \psi\partial_t\psi^*\right), \qquad \vec{J} = -\frac{i}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)\ . \tag{20}$$

   (d) Argue that $\rho$ in (20) cannot be interpreted as probability density.

2. **Commutation relations of annihilation and creation operators (20 points)**

   For the real scalar field theory discussed in lecture, i.e.

   $$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{21}$$

   we showed that the time evolution of quantum operator $\phi(\vec{x},t)$ is given by

   $$\phi(\vec{x},t) = \int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left(a_{\vec{k}}u_{\vec{k}}(\vec{x},t) + a_{\vec{k}}^\dagger u_{\vec{k}}^*(\vec{x},t)\right) \tag{22}$$

where

$$\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}, \qquad u_{\vec{k}}(\vec{x}, t) = e^{-i\omega_{\vec{k}} t + i\vec{k}\cdot\vec{x}} \ . \tag{23}$$

We use $\pi(\vec{x}, t)$ to denote the momentum density conjugate to $\phi$. The canonical commutation relations among $\phi$ and $\pi$ are

$$[\phi(\vec{x}, t), \phi(\vec{x}', t)] = 0 = [\pi(\vec{x}, t), \pi(\vec{x}', t)], \quad [\phi(\vec{x}, t), \pi(\vec{x}', t)] = i\delta^{(3)}(\vec{x} - \vec{x}') \ . \tag{24}$$

(a) Show that it is enough to impose (24) at $t = 0$. In other words, once we impose them at $t = 0$, then the relations at general $t$ are automatically satisfied.

*Note: This statement in fact applies not only to* $V(\phi) = \frac{1}{2}m^2\phi^2$*, but any potential* $V(\phi)$.

(b) Express $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ in terms of $\phi(\vec{k})$ and $\pi(\vec{k})$, where $\phi(\vec{k})$ and $\pi(\vec{k})$ are Fourier transforms of $\phi(\vec{x}, t = 0)$ and $\pi(\vec{x}, t = 0)$, i.e.

$$\phi(\vec{k}) = \int d^3x \, e^{-i\vec{k}\cdot\vec{x}} \phi(\vec{x}, t = 0) \tag{25}$$

and similarly for $\pi$.

(c) Using the expressions you derived in part (b) to deduce the commutations relations

$$[a_{\vec{k}}, a_{\vec{k}'}], \qquad [a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'}], \qquad [a_{\vec{k}}, a^\dagger_{\vec{k}'}] \tag{26}$$

from the commutation relations (24) at $t = 0$.

3. **Expressing Noether charges in terms of creation and annihilation operators (20 points)**

In pset 1 you obtained the conserved charges associated with spacetime translational symmetries for a complex scalar field theory. The results there can be easily converted to the corresponding expressions for a real scalar field theory (21).

(a) Express the Hamiltonian $H$ of (21) in terms of $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$.

(b) Express the conserved charges $P^i$ for spatial translations for (21) in terms of $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$.

(c) Starting with

$$\phi(0, 0) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left( a_{\vec{k}} + a^\dagger_{\vec{k}} \right) \tag{27}$$

show that under the action of translation operators

$$\phi(\vec{x}, t) = e^{iHt - iP^i x^i} \phi(0, 0) e^{-iHt + iP^i x^i} \ . \tag{28}$$

*Note: This problem becomes trivial if you recall the following formula for a harmonic oscillator*

$$e^{i\alpha N} a e^{-i\alpha N} = e^{-i\alpha} a, \qquad N = a^\dagger a \tag{29}$$

and $\alpha$ is a constant.

4. **Noether charges for Lorentz symmetries of the real scalar field theory (20 points + 10 bonus points)**

In this problem we work out the conserved current corresponding to Lorentz symmetries of (21).

(a) Consider an infinitesimal Lorentz transformation (12)–(13). Show that (12) satisfies (6) to first order in $\omega_{\mu\nu}$, so does give a Lorentz transformation.

(b) Write down how $\phi$ transforms under an infinitesimal Lorentz transformation (see (9)) and show that the conserved Noether current for this transformation can be written as

$$J^{\mu\lambda\nu} = x^\lambda T^{\mu\nu} - x^\nu T^{\mu\lambda} \tag{30}$$

where $T^{\mu\nu}$ is the conserved energy-momentum tensor which we have already derived in pset 1.

*Note: this part does not involve complicated calculations. If you find yourself in a massive calculation, pause, and try to find a simpler approach.*

(c) Use the conservation of the energy-momentum tensor to verify that the current (30) is indeed conserved, i.e.

$$\partial_\mu J^{\mu\lambda\nu} = 0 \ . \tag{31}$$

**This problem is complete if you finish the above parts. The part below is an instructive exercise, but is calculation heavy. It is given as a bonus problem (10 extra points) for those of you who would like to have more fun.**

(d) Consider the conserved charges associated with $J^{\mu\lambda\nu}$

$$M^{\lambda\nu} = \int d^3x \, J^{0\lambda\nu} \tag{32}$$

Express the conserved charges $M^{\mu\nu}$ for Lorentz symmetries for (21) in terms of $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$.

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