Quantum Field Theory I (8.323) Spring 2023
Assignment 12

May. 2, 2023

- Please remember to put **your name** at the top of your paper.
- **This is the last pset of the semester, with total points 90+30 (30 are bonus points).**

Readings

- Peskin & Schroeder Chap. 4.5 – 4.6
- Peskin & Schroeder Chap. 4.7–4.8
- Peskin & Schroeder Chap. 5

Notes:

1. For discussion of S-matrix and cross section, you can also read Weinberg Vol1. Sec. 3.1, 3.2 and 3.4. Another good source is Sredicki Vol1. chap. 11, which I partially followed in my discussion.

2. Decay rate: consider the decay process

$$p_1 \to k_1 + k_2 + \cdots + k_n \ . \tag{1}$$

Then the differential decay rate of decaying into n final particles with one particle in the range $d^3\vec{k}_1$ around $\vec{k}_1$, one particle in the range $d^3\vec{k}_2$ around $\vec{k}_2$, and etc., is given by

$$d\Gamma = \frac{1}{2E_1}|M|^2 d\mu \tag{2}$$

where E_1 is the energy of the decaying particle, M is the corresponding scattering amplitude, and $d\mu$ is the Lorentz invariant differential measure

$$d\mu = (2\pi)^4 \delta^{(4)}\left(p_\alpha - p_\beta\right) \prod_{i=1}^{n} \frac{d^3\vec{k}_i}{(2\pi)^3} \frac{1}{2k_i^0} \tag{3}$$

where p_α and p_β denote the total momentum of the initial and final state respectively.

The full decay rate for this process is given by

$$\Gamma = \frac{1}{\Lambda} \int d\Gamma \tag{4}$$

where the integration is over all momenta. $1/\Lambda$ is a numerical factor which is 1 if the final particles are all distinct. If there are a subset of m *identical* particles in the final state, then $\Lambda = m!$ as the integrations over all momenta over-count the phase space.

3. Cross section: consider the scattering process

$$p_1 + p_2 \to k_1 + k_2 + \cdots + k_n \ . \tag{5}$$

Then the differential cross section of finding n final particles with one particle in the range $d^3\vec{k}_1$ around $\vec{k}_1$, one particle in the range $d^3\vec{k}_2$ around $\vec{k}_2$, and etc., is given by

$$d\sigma = \frac{1}{4\Sigma}|M|^2 d\mu, \qquad \Sigma = \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} \tag{6}$$

where M is the corresponding scattering amplitude, and $d\mu$ is again given by (3). m_1, m_2 are the masses of p_1 and p_2 respectively. The total cross section is given by

$$\sigma_{\rm tot} = \frac{1}{\Lambda} \int d\sigma \tag{7}$$

where as in (4) $1/\Lambda$ is a numerical factor which accounts for the over-counting of phase space if there are identical particles in the final state.

4. It is often convenient to introduce

$$s = -(p_1 + p_2)^2 \tag{8}$$

which is the invariant mass square for the whole system.

5. It is often convenient to consider the center of mass frame where

$$\vec{p}_1 = -\vec{p}_2 \equiv \vec{p}_{cm} \ . \tag{9}$$

The magnitude $|\vec{p}_{CM}|$ can be expressed in terms of s by solving the equation

$$\sqrt{s} = \sqrt{\vec{p}_{cm}^2 + m_1^2} + \sqrt{\vec{p}_{cm}^2 + m_2^2} \ . \tag{10}$$

Equation (6) can also be written as

$$d\sigma = \frac{|M|^2}{4|\vec{p}_{cm}|\sqrt{s}} d\mu \ . \tag{11}$$

6. Two-to-two scattering: for two-to-two scattering

$$p_1 + p_2 \to k_1 + k_2 \tag{12}$$

in addition to (8) it is convenient to introduce

$$t \equiv -(p_1 - k_1)^2, \qquad u = -(p_1 - k_2)^2 \,. \tag{13}$$

Note that

$$s + t + u = m_1^2 + m_2^2 + m_1'^2 + m_2'^2 \tag{14}$$

where m_1' and m_2' are the masses of k_1 and k_2 respectively. In the center of mass frame, from momentum conservation, the final spatial momenta satisfy

$$\vec{k}_1 = -\vec{k}_2 \equiv \vec{k}_{cm} \tag{15}$$

where $|\vec{k}_{cm}|$ can again be expressed in terms of s.

The differential cross section can be further simplified and written as

$$d\sigma = \frac{|M|^2}{64\pi^2 s} \frac{|\vec{k}_{cm}|}{|\vec{p}_{cm}|} d\Omega_{cm} \tag{16}$$

where $d\Omega_{cm}$ denotes the angular part of $d^3\vec{k}_{cm}$.

Problem Set 12

1. **Decay of a scalar particle (20 points)**

   Consider a theory with two scalar fields ϕ and χ

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2}M^2\phi^2 - \tfrac{1}{2}(\partial_\mu \chi)^2 - \tfrac{1}{2}m^2\chi^2 + \tfrac{1}{2}\, g\phi\chi^2 \,. \tag{17}$$

   Assume $M > 2m$ and the coupling constant g is small. Calculate the decay rate Γ of ϕ-particles to the lowest order in g.

2. **Cross-sections (20 points)**

   In this problem we consider a toy model of the process in which an electron-positron collision produces a quark-antiquark final state through an intermediate photon: $e^+e^- \to \gamma \to q\,\bar{q}$. The role of the electron and that of the positron is played by a massless scalar field ψ. The photon is represented by a massless scalar field ϕ. Finally, the quarks are represented by a massive scalar field χ. The relevant Lagrangian density is

$$\mathcal{L} = -\frac{1}{2}(\partial\psi)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}m^2\chi^2 + \frac{1}{2}g'\phi\psi^2 + \frac{1}{2}g\phi\chi^2 \,. \tag{18}$$

   Assume that couplings g and g' are small and of comparable magnitude.

(a) Consider the process:

$$\psi\,\psi \to \chi\,\chi\ . \tag{19}$$

Compute the total cross section to lowest order in couplings. Express your answer in terms of s, t, u variables.

(b) Consider the process:

$$\psi\,\psi \to \psi\,\psi\ . \tag{20}$$

Compute the differential cross section to lowest order in couplings. Express your answer in terms of s, t, u variables .

3. **Rutherford scattering (30 + 10 points)**

   (a)–(c) Peskin & Schroeder: Prob. 4.4.

   The following part is counted as bonus (10 points)

   (d) Repeat the calculation of part (c) in the relativistic regime. After averaging over the spin of the initial state and summing over the spin of the final state, show that the cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4|\vec{p}|^2\beta^2\sin^2\frac{\theta}{2}}\left(1 - \beta^2\sin^2\frac{\theta}{2}\right) \tag{21}$$

   where $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant, $\vec{p}$ is the spatial momentum of the electron, and β is its velocity.

4. **Electron-Muon scattering (20+20 points)**

   Consider the scattering process

$$e^- + \mu^- \to e^- + \mu^-\ . \tag{22}$$

   Electron and muon have mass m_e and m_μ respectively.

   (a) Calculate the scattering amplitude $\mathcal{M}$.

   (b) Calculate

$$\frac{1}{4}\sum_{\text{spin}} |\mathcal{M}|^2 \tag{23}$$

   where $\sum_{\text{spin}}$ denotes average over the spins of the initial state and sum over the spins of the final state. Express your answer in terms of s, t, u variables defined in (8) and (13).

   The following two parts are bonus parts (20 points)

   (c) Calculate the differential cross section in the center of mass frame.

(d) Obtain the differential cross section in the muon rest frame. Show that in the limit $m_\mu \to \infty$, one recovers the cross section of Rutherford scattering, equation (21).

[Hint: you can consult Sec. 5.4 of Peskin and Schroeder which contains all the key results. But you need to work out some of the intermediate steps yourself.]

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