Class 3 in-class problems, 18.05, Spring 2022

Concept questions

Concept question 1.
Toss a coin 4 times. Let A = 'at least three heads' and B = 'first toss is tails'.

1. What is $P(A|B)$?

(a) 1/16 (b) 1/8 (c) 1/4 (d) 1/5

2. What is $P(B|A)$?

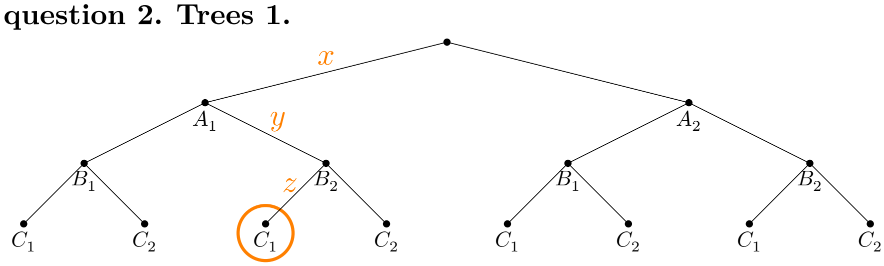
(a) 1/16 (b) 1/8 (c) 1/4 (d) 1/5

Solution: 1. (b) 1/8. 2. (d) 1/5.

Counting we find $|A| = 5$, $|B| = 8$ and $|A \cap B| = 1$. Since all sequences are equally likely

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{|A \cap B|}{|B|} = 1/8. \quad P(B|A) = \frac{|B \cap A|}{|A|} = 1/5.$$

Concept question 2. Trees 1.



1. The probability x represents

(a) $P(A_1)$ (b) $P(A_1|B_2)$ (c) $P(B_2|A_1)$ (d) $P(C_1|B_2 \cap A_1)$.

Solution: (a) $P(A_1)$.

Concept question 3. Trees 2.
2. The probability y represents

(a) $P(B_2)$ (b) $P(A_1|B_2)$ (c) $P(B_2|A_1)$ (d) $P(C_1|B_2 \cap A_1)$.

Solution: (c) $P(B_2|A_1)$.

Concept question 4. Trees 3.
3. The probability z represents

(a) $P(C_1)$ (b) $P(B_2|C_1)$ (c) $P(C_1|B_2)$ (d) $P(C_1|B_2 \cap A_1)$.

Solution: (d) $P(C_1|B_2 \cap A_1)$.

Concept question 5. Trees 4.
4. The circled node represents the event

(a) C_1 (b) $B_2 \cap C_1$ (c) $A_1 \cap B_2 \cap C_1$ (d) $C_1|B_2 \cap A_1$.

Solution: (c) $A_1 \cap B_2 \cap C_1$.

In class examples

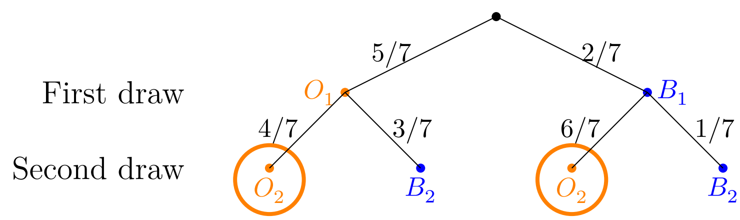
Class example 1.

- *Organize computations*
- *Compute total probability*
- *Compute Bayes' formula*

Example. *Game: 5 orange and 2 blue balls in an urn. A random ball is selected and replaced by a ball of the other color; then a second ball is drawn.*

1. What is the probability the second ball is orange?

2. What is the probability the first ball was orange given the second ball was orange?



Solution: 1. Let O_1 be the event the first ball is orange. Likewise for O_2, B_1, B_2. The law of total probability gives $P(O_2) = \frac{5}{7}\cdot\frac{4}{7} + \frac{2}{7}\cdot\frac{6}{7} = \frac{32}{49}$

2. Bayes' rule gives $P(O_1|O_2) = \frac{P(O_1 \cap O_2)}{P(O_2)} = \frac{20/49}{32/49} = \frac{20}{32}$

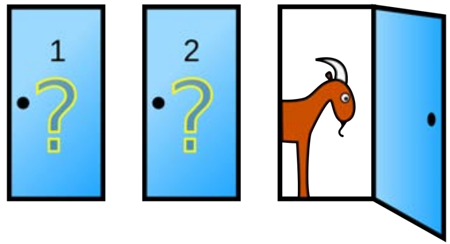
Board questions

Problem 1. Monty Hall

- *One door hides a car, two hide goats.*
- *The contestant chooses any door.*
- *Monty always opens a different door with a goat. (He can do this because he knows where the car is.)*
- *The contestant is then allowed to switch doors if they want.*

What is the best strategy for winning a car?

(a) Switch (b) Don't switch (c) It doesn't matter



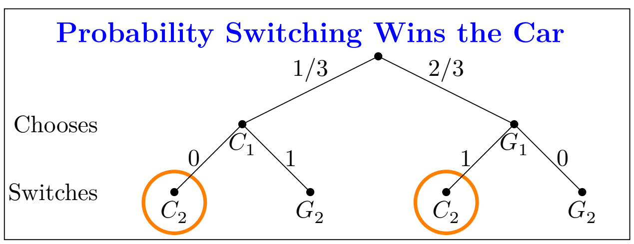
Organize the Monty Hall problem into a tree and compute the probability of winning if you always switch.

Hint first break the game into a sequence of actions.

Solution: Let P_{switch} be the probability function when the contestant uses the switching strategy. Let C represent a car and G a goat.

We will see that $P_{\text{switch}}(C) = 2/3$

One way to show this is with a tree representing the switching strategy: First the contestant chooses a door, (then Monty shows a goat), then the contestant switches doors.



The (total) probability of C is $P_{\text{switch}}(C) = \frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 1 = \frac{2}{3}$.

Writing this in symbols using the law of total probability: Here G_1 represents a goat is behind the original door chosen and G_2 means a goat is behind the door that is switched to. Likewise C_1 and C_2.

$$P_{\text{switch}}(C_2) = P_{\text{switch}}(C_2|C_1)P(C_1) + P_{\text{switch}}(C_2|G_1)P(G_1) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{2}{3}.$$

Problem 2. Independence

Roll two dice and consider the following events

- A = *'first die is 3'*
- B = *'sum is 6'*
- C = *'sum is 7'*

A is independent of

(a) B and C (b) B alone (c) C alone (d) Neither B or C.

Solution: (c). *(Explanation below)*

$P(A) = 1/6$, $P(A|B) = 1/5$. Not equal, so not independent.

$P(A) = 1/6$, $P(A|C) = 1/6$. Equal, so independent.

Notice that knowing B, removes 6 as a possibility for the first die and makes A more probable. So, knowing B occurred changes the probability of A.

But, knowing C does not change the probabilities for the possible values of the first roll; they are still 1/6 for each value. In particular, knowing C occured does not change the probability of A.

We could also have done this problem by showing

$$P(B|A) \neq P(B) \text{ or } P(A \cap B) \neq P(A)P(B).$$

Problem 3. Evil Squirrels

Of the one million squirrels on MIT's campus most are good-natured. But one hundred of them are pure evil! An enterprising student in Course 6 develops an "Evil Squirrel Alarm" which they offer to sell to MIT for a passing grade. MIT decides to test the reliability of the alarm by conducting trials.

- *When presented with an evil squirrel, the alarm goes off 99% of the time.*
- *When presented with a good-natured squirrel, the alarm goes off 1% of the time.*

(a) If a squirrel sets off the alarm, what is the probability that it is evil?

(b) Should MIT co-opt the patent rights and employ the system?

One solution

(This is a base rate fallacy problem)

We are given:

$$P(\text{nice}) = 0.9999, P(\text{evil}) = 0.0001 \text{ (base rate)}$$
$$P(\text{alarm} \mid \text{nice}) = 0.01, P(\text{alarm} \mid \text{evil}) = 0.99$$

$$\begin{aligned} P(\text{evil} \,|\, \text{alarm}) &= \frac{P(\text{alarm} \,|\, \text{evil}) P(\text{evil})}{P(\text{alarm})} \\ &= \frac{P(\text{alarm} \,|\, \text{evil}) P(\text{evil})}{P(\text{alarm} \,|\, \text{evil}) P(\text{evil}) + P(\text{alarm} \,|\, \text{nice}) P(\text{nice})} \\ &= \frac{(0.99)(0.0001)}{(0.99)(0.0001) + (0.01)(0.9999)} \\ &\approx 0.01 \end{aligned}$$

Summary:

Probability a random test is correct =	0.99
Probability a positive test is correct ≈	0.01

These probabilities are not the same!

Alternative method of calculation:

	Evil	Nice	
Alarm	99	9999	10098
No alarm	1	989901	989902
	100	999900	1000000

Solution: (a) This is the same solution as above, but in a more compact notation. Let E be the event that a squirrel is evil. Let A be the event that the alarm goes off. By Bayes' Theorem, we have:

$$\begin{aligned} P(E \,|\, A) &= \frac{P(A \,|\, E) P(E)}{P(A \,|\, E) P(E) + P(A \,|\, E^c) P(E^c)} \\ &= \frac{0.99 \frac{100}{1000000}}{(0.99) \cdot \left(\frac{100}{1000000}\right) + (0.01) \cdot \left(\frac{999900}{1000000}\right)} \\ &\approx 0.01. \end{aligned}$$

(b) No. The alarm would be more trouble than its worth, since for every true positive there are about 99 false positives.

Problem 4. Dice Game

1. *The Randomizer holds the 6-sided die in one fist and the 8-sided die in the other.*
2. *The Roller selects one of the Randomizer's fists and covertly takes the die.*
3. *The Roller rolls the die in secret and reports the result to the table.*

Given the reported number, what is the probability that the 6-sided die was chosen? (Find the probability for each possible reported number.)

Solution: If the number rolled is 1-6 then $P(\text{six-sided}) = 4/7$.

If the number rolled is 7 or 8 then $P(\text{six-sided}) = 0$.

This is a Bayes' formula problem. For concreteness let's suppose the roll was a 4. What we want to compute is $P(\text{6-sided}|\text{roll 4})$. But, what is easy to compute is $P(\text{roll 4}|\text{6-sided})$.

Bayes' formula says

$$\begin{aligned} P(\text{6-sided}|\text{roll 4}) &= \frac{P(\text{roll 4}|\text{6-sided})P(\text{6-sided})}{P(4)} \\ &= \frac{(1/6)(1/2)}{(1/6)(1/2) + (1/8)(1/2)} = 4/7. \end{aligned}$$

The denominator is computed using the law of total probability:

$$P(4) = P(4|\text{6-sided})P(\text{6-sided}) + P(4|\text{8-sided})P(\text{8-sided}) = \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{8} \cdot \frac{1}{2}.$$

Note that any roll of 1,2,...6 would give the same result. A roll of 7 (or 8) would give clearly give probability 0. This is seen in Bayes' formula because the term $P(\text{roll 7}|\text{6-sided}) = 0$.