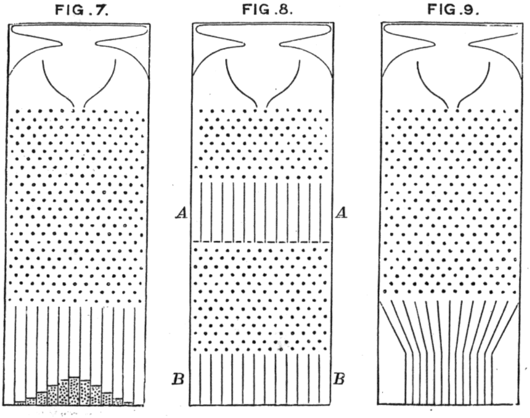
Discrete Random Variables; Expectation
18.05 Spring 2022



https://en.wikipedia.org/wiki/Bean_machine#/media/File:Quincunx_(Galton_Box)_-_Galton_1889_diagram.png

https://www.youtube.com/watch?v=9xUBhhM4vbM

Announcements/Agenda

- **Office hours today:** Gabriel, after class in this room.
- Evil squirrels from last time
- Review of reading
  - random variable
  - probability mass function
  - cumulative distribution function
  - expected value
- Named distributions: Bernoulli, binomial, geometric

Reading Review

Random variable X: assigns a number to each outcome, i.e.

$$X : S \to \mathbf{R}$$

"$X = a$" denotes the event $\{\omega \,|\, X(\omega) = a\}$.

Probability mass function (pmf) of X is given by

$$p(a) = P(X = a).$$

Cumulative distribution function (cdf) of X is given by

$$F(a) = P(X \leq a).$$

Example

We play a game where you roll a 4-sided die. The payoff is

-\$2 for a 1, -\$1 for a 2, \$0 for a 3, \$4 for a 4.

Let X be the payoff function. It is a random variable with the following table.

outcome ω:	1	2	3	4
values of X:	-2	-1	0	4
pmf $p(a)$:	1/4	1/4	1/4	1/4
cdf $F(a)$:	1/4	2/4	3/4	4/4

The cdf is the probability 'accumulated' from the left.

Question: What are $F(-1)$, $F(0.5)$, $F(-5)$, $F(5)$, $p(-1)$, $p(0.5)$?

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Question: What are $F(-1)$, $F(0.5)$, $F(-5)$, $F(5)$, $p(-1)$, $p(0.5)$?

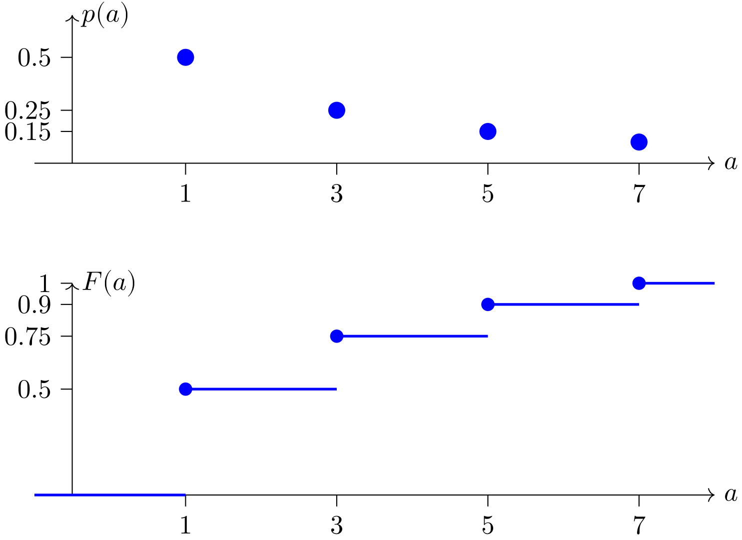
Answer: $F(-1) = 2/4$, $F(0.5) = 3/4$, $F(-5) = 0$, $F(5) = 1$, $p(-1) = 1/4$, $p(0.5) = 0$.

Properties of PMF and CDF

1. $0 \leq p(a) \leq 1$
2. $p(a)$ sums to 1.

1. $F(a)$ is nondecreasing
2. Way to the left, i.e. as $a \to -\infty$, $F(a)$ is 0
3. Way to the right, i.e. as $a \to \infty$, $F(a)$ is 1.

Plots of PMF and CDF



Concept Questions: cdf and pmf

Problem 1. Suppose X a random variable with the CDF shown.

values of X:	1	3	5	7
cdf $F(a)$:	0.5	0.75	0.9	1

What is $P(X \leq 3)$?

(a) 0.15 (b) 0.25 (c) 0.5 (d) 0.75

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Problem 2. What is $P(X = 3)$?
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Meaning of Expected Value

Example. Suppose an experiment produces random values 80, 1, 10 with the following table

value:	80	1	10
pmf:	1/5	1/5	3/5

What is the *expected* average value over many experiments?

Solution: Say 5 experiments: 80,80,10,10,10. Ave. $= 190/5 = 38$

In 5 experiments, average will vary a lot. What about in 500?

Expect each number occurs roughly in proportion to its probability. Suppose this is exactly what happens and compute the average.

value:	80	1	10
expected counts:	100	100	300

$$\text{average} = \frac{100 \cdot 80 + 100 \cdot 1 + 300 \cdot 10}{500} = \frac{1}{5} \cdot 80 + \frac{1}{5} \cdot 1 + \frac{3}{5} \cdot 10 = \boxed{22.2}$$

This is the 'expected average' = expected value or expectation.

Expected Value

Random variable X: Takes values $x_1, x_2, \ldots, x_n$, has pmf $p(x_i)$.
The **expected value** of X is defined by

$$E[X] = p(x_1)x_1 + p(x_2)x_2 + \ldots + p(x_n)x_n = \sum_{i=1}^{n} p(x_i)\, x_i$$

- It is a weighted average.
- In statistics it is called a measure of central tendency.

Properties of $E[X]$

- $E[X+Y] = E[X] + E[Y]$ (linearity I)
- $E[aX+b] = aE[X] + b$ (linearity II)
- $E[h(X)] = \sum_i h(x_i)\, p(x_i)$

Expectation Examples

Example 1. Find $E[X]$.

1.	X:	3	4	5	6
2.	pmf:	1/4	1/2	1/8	1/8

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Example 2. Suppose $X \sim$ Bernoulli(p). Find $E[X]$.

1.	X:	0	1
2.	pmf:	$1-p$	p

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Example 3. Suppose $X = X_1 + X_2 + ... + X_{12}$, where $E[X_i] = 0.25$ for each i. Find $E[X]$.

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Example 3. Suppose $X = X_1 + X_2 + ... + X_{12}$, where $E[X_i] = 0.25$ for each i. Find $E[X]$.

$$E[X] = E[X_1] + E[X_2] + ... E[X_{12}] = 12 \cdot (0.25) = 3$$

(Linearity of expectation.)

Expectation Example 4

Example 4. Consider the random variable X with the following table. Compute $E[X]$ and $E[X^2 + X]$

1.	X:	-2	-1	0	1	2
2.	pmf:	$1/5$	$1/5$	$1/5$	$1/5$	$1/5$

Expectation Example 4

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1. X: -2 -1 0 1 2
2. pmf: 1/5 1/5 1/5 1/5 1/5

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3. $E[X]$ = -2/5 - 1/5 + 0/5 + 1/5 + 2/5 = 0

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1.	X:	-2	-1	0	1	2
2.	pmf:	1/5	1/5	1/5	1/5	1/5
3.	$E[X] =$ -2/5 - 1/5 + 0/5 + 1/5 + 2/5 $= 0$					
4.	$X^2 + X$:	2	0	0	2	6
5.	$E[X^2 + X] = 2/5 + 0/5 + 0/5 + 2/5 + 6/5 = 2$					

Line 3 computes $E[X]$ by multiplying the probabilities in line 2 by the values in line 1 and summing.

Line 4 gives the values of $X^2 + X$.

Line 5 computes $E[X^2 + X]$ by multiplying the probabilities in line 2 by the values in line 4 and summing. This illustrates the use of the formula

$$E[h(X)] = \sum_i h(x_i)\, p(x_i).$$

Board Question: Computing expectation

1. Suppose X is a random variable with the following pmf.

X:	1	2	3
pmf:	1/4	1/2	1/4

Find $E[X]$ and $E[1/X]$.

Board Question: Interpreting expectation

2. **(a)** Would you accept a gamble that offers a 10% chance to win $\$95$ and a 90% chance of losing $\$5$?

(b) Would you pay $\$5$ to participate in a lottery that offers a 10% percent chance to win $\$100$ and a 90% chance to win nothing?

Hint: find the expected value of your winnings in each case.

The solutions contain some extended remarks about framing bias and tolerance for risk.

Board Question: Musical chairs or linearity of expectation

3. Suppose that there are n people at your table and everyone got up, ran around the room, and sat back down randomly (i.e., all seating arrangements are equally likely).

What is the expected value of the number of people sitting in their original seat?

(We will explore this with simulations in Friday Studio.)

Deluge of discrete distributions: Bernoulli(p)

Bernoulli(p): binary choice; $p = P(1)$

Values:	1 (for success) 0 (for failure)
PMF:	$p(1) = p, \;\; p(0) = 1 - p$
Expectation:	$X \sim \text{Bernoulli}(p) \;\Rightarrow\; E[X] = p$

In more neutral language: One toss of a biased coin.

Values: 1 (for heads) 0 (for tails)

Probability table

X:	0	1
pmf:	$1 - p$	p

Deluge of discrete distributions: Binomial(n, p)

$X \sim$Binomial$(n,p) = \#$ of successes in n independent Bernoulli(p) trials

Values: 0, 1, 2, ..., n

PMF: $p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

Expectation: $X \sim \text{Binomial}(n, p) \;\Rightarrow\; E[X] = np$

In more neutral language: Number of heads in n tosses of a biased coin.

Deluge of discrete distributions: Geometric(p)

$X \sim$ Geometric(p) = # of failures before first success in a sequence of independent Bernoulli(p) trials

Values: 0, 1, 2, 3, ...

PMF: $p(k) = p(1-p)^k$

Expectation: $X \sim \text{Geometric}(p) \;\Rightarrow\; E[X] = (1-p)/p$.

In more neutral language: Number of tails before the first heads from repeated tosses of a coin.

Neutral language helps avoid confusion over whether we want the number of successes before the first failure or vice versa.

Board Questions

4. **(a)** Suppose $X \sim$ Bernoulli(p). Find $E[X]$.

(This is important! Remember it!)

(b) Suppose $Y = X_1 + X_2 + ... + X_{12}$, where each $X_i \sim$ Bernoulli(0.25). Find $E[Y]$.

5. (Don't let one failure stop you!)
Let $X = \#$ of successes before the *second* failure of a sequence of independent Bernoulli(p) trials. Find the pmf of X.

Hint: this requires a small amount of counting.

Fiction

Gambler's fallacy: [roulette] if black comes up several times in a row then the next spin is more likely to be red.

Hot hand: NBA players get 'hot'.

Fact

P(red) remains the same.

The roulette wheel spins are independent. (Monte Carlo, 1913).

The data show that player who has made 5 shots in a row is no more likely than usual to make the next shot.

(Currently, there seems to be some disagreement about this.)

More discussion and references given in the solutions to today's problems.

Amnesia

Show that $Geometric(p)$ is memoryless, i.e.

$$P(X = n + k \,|\, X \geq n) = P(X = k)$$

Explain why we call this memoryless.