Quantum Field Theory I (8.323) Spring 2023
Assignment 6

Mar. 14, 2023

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder Chap. 4.1 – 4.6
- Peskin & Schroeder Chap. 9.1–9.2

Notes:

1. We showed that for a quantum field theory of real scalar ϕ, n-point time-ordered correlation functions can be expressed in terms of path integrals as

$$\begin{aligned} G_n(x_1, \cdots, x_n) &= \langle \Omega | T(\phi(x_1) \cdots \phi(x_n)) | \Omega \rangle \\ &= \frac{\int D\phi \, e^{iS[\phi]} \phi(x_1) \cdots \phi(x_n)}{\int D\phi \, e^{iS[\phi]}} \end{aligned} \tag{1}$$

   where the path integrals of both upstairs and downstairs are defined by integrating over all configurations of $\phi(t, \vec{x})$ satisfying

$$\lim_{t \to \pm\infty} \phi(t, \vec{x}) = 0, \quad \lim_{|\vec{x}| \to \infty} \phi(t, \vec{x}) = 0 \ . \tag{2}$$

2. In perturbation theory, it is convenient to express G_n in terms of free theory quantities as

$$G_n(x_1, \cdots x_n) = \frac{\left\langle 0 \middle| T\phi(x_1)\phi(x_2) \cdots \phi(x_n) e^{-i \int_{-\infty}^{\infty} H_I dt} \middle| 0 \right\rangle}{\left\langle 0 \middle| T e^{-i \int_{-\infty}^{\infty} H_I dt} \middle| 0 \right\rangle} \tag{3}$$

   Note that

$$\int d^4x \, \mathcal{L}_I = -\int dt \, H_I \tag{4}$$

   and we will use both these two forms.

3. When we write expressions like

$$\left\langle 0 \left| T\phi(x_1)\phi(x_2)e^{-i\int_{-\infty}^{\infty} H_I dt} \right| 0 \right\rangle \tag{5}$$

the time ordering includes both $\phi(x_1)\phi(x_2)$ and $e^{-i\int_{-\infty}^{\infty} H_I dt}$. One might be puzzled by what we mean by time ordering of $e^{-i\int_{-\infty}^{\infty} H_I dt}$ as there is a time integral in the exponential. One should understand $e^{-i\int_{-\infty}^{\infty} H_I dt}$ as a power series and then do time ordering inside the integrals. More explicitly,

$$Te^{-i\int_{-\infty}^{\infty} H_I dt} \equiv \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 dt_2 \cdots dt_n \, T\left(H_I(t_1)H_I(t_2)\cdots H_I(t_n)\right) \ . \tag{6}$$

4. Peskin and Schroeder's notion of connected diagrams, as discussed on p.97-p.98, is confusing and non-standard, as it includes diagrams with disconnected parts! To make things even more confusing, they use different notions of connected diagrams in Sec. 4.4 and Sec. 4.6. The definition of Sec. 4.6 (see p.114) is the same as that in lecture.

   Below by **connected diagrams** we mean diagrams with no disconnected parts. For example, among diagrams in eq (4.58) on p.99 of Peskin and Schroeder, only the first diagram in the second line and the diagrams in the third line are connected.

5. Some of you may have observed that some expressions for correlation functions I wrote down in lectures are divergent. You will also find divergences in various expressions you will write down in this pset. As is the case for the energy density of the free scalar field theory we already encountered, such divergences are ubiquitous in quantum field theories and reflect that a QFT has an infinite number of *local* degrees of freedom. We then have to understand two issues:

   (a) How to make sense of physical observables which are seemingly divergent.

   (b) Why we can do perturbation theory if the terms in perturbative series are divergent.

   The first issue is addressed by the program of renormalization, which is a mathematical procedure for extracting unambiguous, finite answers out of apparently divergent expressions.

   The renormalization program, developed in late 1940's and early 1950's, was hugely successful in getting sensible answers which compared well with experiments. It, however, does not address the second issue.

   The second issue was only understood after the development of Wilson's renormalization group program in late 1960's and early 1970's.

Both these issues will be discussed in QFT2. In QFT1, we will only concern ourselves with the leading term in perturbative expansions of scattering amplitudes, which as we will see are not divergent.

Problem Set 6

1. **Particle production by an external source: continued (10 points)**

   Consider again Prob. 2 of Pset 4. Introduce

$$Z[J] = \int D\phi \, e^{i\int d^4x \, \mathcal{L}}, \quad Z_0 = Z[J=0] = \int D\phi \, e^{i\int d^4x \, \mathcal{L}_0} \tag{7}$$

   where $\mathcal{L}$ is given by equation (17) of pset 4, which we copy here for your convenience

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J(x)\phi = \mathcal{L}_0 + J(x)\phi \ . \tag{8}$$

   Using similar tricks as we introduced in lecture to derive (1), you can show that

$$\langle 0, +\infty | 0, -\infty \rangle = \frac{Z[J]}{Z_0} \tag{9}$$

   with an appropriate $i\epsilon$ prescription. I will not ask you to derive the above equation, but you should think about yourself how it can be derived.

   Using (9) to find the probability of no particle production

$$P_0 = |\langle 0, +\infty | 0, -\infty \rangle|^2 \ . \tag{10}$$

   by directly evaluating the path integrals (7). You should verify that it reproduces the answer you obtained in part 2(h) of pset 4.

2. **Connected diagrams (30 points)**

   Below the notion of connected diagrams follows that defined in item 4 of Notes at the beginning.

   Consider the $\lambda\phi^4$ theory discussed in lecture, i.e.

$$H_I = \frac{\lambda}{4!}\int d^3x \, \phi^4(x) \ . \tag{11}$$

(a) List all *connected* diagrams for

$$\left\langle 0|T\phi(x_1)\phi(x_2)e^{-i\int_{-\infty}^{\infty}H_I dt}|0\right\rangle \tag{12}$$

to order $O(\lambda^2)$, and give the symmetry factor for each diagram.

For diagrams at $O(\lambda^0)$ and $O(\lambda)$ orders write down their expressions in *both* coordinate and momentum space.

At order $O(\lambda^2)$ choose one diagram to write down its coordinate and momentum space expressions.

You do not need to evaluate the integrals.

(b) List all *connected* diagrams of the four-point function

$$G_4(x_1, x_2, x_3, x_4) = \langle\Omega|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\Omega\rangle \tag{13}$$

to order $O(\lambda^2)$, and choose *two of them* to write down their respective expressions in *both* coordinate and momentum space. You do not need to do the integrals.

3. **Vacuum diagrams (30 points)**

For $\lambda\phi^4$ theory (11), consider the quantity

$$Z_0 = \left\langle 0\middle|Te^{-i\int_{-\infty}^{\infty}H_I dt}\middle|0\right\rangle \tag{14}$$

where the expectation value is evaluated in the free theory. We also assume that the free theory vacuum $|0\rangle$ is properly normalized, i.e. $\langle 0|0\rangle = 1$.

(a) Consider

$$W_0 = \log Z_0\ . \tag{15}$$

Show that W_0 can be written in a form

$$W_0 = C - i\varepsilon VT \tag{16}$$

where C is some constant independent of the spacetime volume, ε is the energy density difference between the full and free theories, and VT is the total *spacetime* volume. Note $T = t - t'$ where $t \to \infty$ and $t' \to -\infty$ are respectively upper and lower limits of the path integrals.

(b) The Feynman diagrams in perturbative expansion of Z_0 have no external lines, which are often called vacuum diagrams or vacuum bubbles. We thus say that Z_0 is obtained by summing over vacuum diagrams. Show that W_0 is the sum of *connected* vacuum diagrams.

Note that the statement in fact applies to all theories.

Hint: consult discussion of Peskin and Schroeder on p.96-98. The discussion of Peskin and Schroeder is not great, but hopefully you can get the key idea and write down your own proof.

(c) Write down the expression for ε to order $O(\lambda^2)$. You can write it either in coordinate or momentum space. It is enough to write it formally as integrations of Feynman propagators. You do not need to evaluate the integrals.

Note: the expression is divergent!

4. **General n-point function (10 points)**

Prove that in evaluating n-point function (3), diagrams that contain factor(s) of vacuum diagrams all cancel. That is, G_n is obtained by summing all diagrams without any vacuum diagram factors. In lecture we saw a specific example of this cancellation when considering two-point function at order $O(\lambda)$. Here you will see that the cancellation is not an accident.

The statement is in fact is true for any H_I. But it is enough if you could prove it for the $\lambda\phi^4$ theory.

Hint: Peskin and Schroeder gives a proof for the two-point function on p.96-98, which can be easily generalized to n-point functions. The discussion of Peskin and Schroeder is not great, but hopefully you can get the key idea and write down your own proof.

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Spring 2023