8.323 Problem Set 8 Solutions

April 11, 2023

Question 1: Proofs of Spinor Identities (21 points)

(a) From the definition

$$\Lambda^\mu{}_\nu = \left(e^{-\frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}}\right)^\mu{}_\nu, \qquad S(\Lambda) = e^{-\frac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma}}$$

Prove the identity

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu \gamma^\nu$$

Hint: use the identity

$$[\Sigma^{\rho\sigma}, \gamma^\mu] = -(\mathcal{J}^{\rho\sigma})^\mu{}_\nu \gamma^\nu$$

By the Baker-Campbell-Hausdorff formula we have

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} L^n(A, B), \qquad L^n(A, B) := \underbrace{[A, \ldots, [A, [A, B]] \ldots]}_{n \text{ times}}$$

We wish to use this with $A = -\frac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma}$ and $B = \gamma^\mu$. Following the hint, we use the identity

$$[\Sigma^{\rho\sigma}, \gamma^\mu] = -(\mathcal{J}^{\rho\sigma})^\mu{}_\nu \gamma^\nu$$

We now compute

$$\begin{aligned} L^n(A, B) &= L^{n-1}(A, [A, B]) = L^{n-1}\left(A, \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu\right) = \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu L^{n-1}(A, \gamma^\nu) \\ &= \left[\frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\right]\left[\frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\nu{}_\lambda\right] L^{n-2}(A, \gamma^\lambda) = \cdots = \left[\left(\frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}\right)^n\right]^\mu{}_\nu \gamma^\nu \end{aligned}$$

In the last expression the power is taken in Lorentz space, i.e. $[(\omega\cdot\mathcal{J})^n]^\mu{}_\nu = (\omega\cdot\mathcal{J})^\mu{}_\alpha(\omega\cdot\mathcal{J})^\alpha{}_\beta \cdots (\omega\cdot\mathcal{J})^\lambda{}_\nu$. Substituting this into the BCH formula, we have

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\left(\frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}\right)^n\right]^\mu{}_\nu \gamma^\nu = \left(e^{+\frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}}\right)^\mu{}_\nu \gamma^\nu = (\Lambda^{-1})^\mu{}_\nu\gamma^\nu$$

(b) Prove the identity

$$S^\dagger = -\gamma^0 S^{-1}\gamma^0$$

From the definition, we start with

$$S(\Lambda)^\dagger = \exp\left(\frac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma\dagger}\right)$$

Now we use the conjugation of the gamma matrices, and $(\gamma^0)^2 = -1$.

$$\Sigma^{\mu\nu\dagger} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]^\dagger = -\frac{i}{4}[\gamma^{\nu\dagger}, \gamma^{\mu\dagger}] = -\frac{i}{4}[\gamma^0\gamma^\nu\gamma^0, \gamma^0\gamma^\mu\gamma^0] = -\frac{i}{4}\gamma^0[\gamma^\mu, \gamma^\nu]\gamma^0 = -\gamma^0\Sigma^{\mu\nu}\gamma^0$$

Substituting this in the exponential, we have

$$\begin{aligned} S(\Lambda)^\dagger &= \exp\left(-\frac{i}{2}\omega_{\rho\sigma}\gamma^0\Sigma^{\rho\sigma}\gamma^0\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i}{2}\omega_{\rho\sigma}\gamma^0\Sigma^{\rho\sigma}\gamma^0\right)^n \\ &= \gamma^0\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma}\right)^n(-1)^{n-1}\gamma^0 = -\gamma^0\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma}\right)^n\gamma^0 = -\gamma^0 S^{-1}(\Lambda)\gamma^0 \end{aligned}$$

In the third equality we use that $(\gamma^0)^2 = -1$ exactly $n-1$ times to pick up a factor of $(-1)^{n-1}$.

(c) From the Lorentz transformation of ψ, show that $\bar{\psi}\psi$ and $\bar{\psi}\gamma^\mu\psi$ transform as a scalar and vector.
We know that under a Lorentz transformation,

$$\psi_\alpha(x) \to \psi'_\alpha(x) = S_\alpha{}^\beta(\Lambda)\psi_\beta(\Lambda^{-1}x) = S(\Lambda)\psi(\Lambda^{-1}x)$$

Using the result from (b), the Dirac conjugate transforms as

$$\bar{\psi}(x) \to \psi'^\dagger(x)\gamma^0 = \psi^\dagger(\Lambda^{-1}x)S(\Lambda)^\dagger\gamma^0 = \psi^\dagger(\Lambda^{-1}x)(-\gamma^0 S(\Lambda)^{-1}\gamma^0)\gamma^0 = \bar{\psi}(\Lambda^{-1}x)S(\Lambda)^{-1}$$

Now we can compute

$$\begin{aligned} \bar{\psi}\psi(x) &\to \bar{\psi}S(\Lambda)^{-1}S(\Lambda)\psi(\Lambda^{-1}x) = \bar{\psi}\psi(\Lambda^{-1}x) \\ \bar{\psi}\gamma^\mu\psi(x) &\to \bar{\psi}S(\Lambda)^{-1}\gamma^\mu S(\Lambda)\psi(\Lambda^{-1}x) = \bar{\psi}S(\Lambda^{-1})\gamma^\mu S^{-1}(\Lambda^{-1})\psi(\Lambda^{-1}x) = \Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\psi(\Lambda^{-1}x) \end{aligned}$$

where in the last equality of the second line we use the result from (a). These are the transformation laws for a scalar and vector.

Question 2: More Spinor Identities (8 points)

Without using any explicit form uf u_s and v_s, show either one of

$$u_r^\dagger(\mathbf{k})u_s(\mathbf{k}) = 2E\delta_{rs}, \qquad v_r^\dagger(\mathbf{k})v_s(\mathbf{k}) = 2E\delta_{rs}$$

In other words, show one of the following.

$$u_r^\dagger(\mathbf{k})u_s(\mathbf{k}) = -\frac{iE}{m}\bar{u}_r(\mathbf{k})u_s(\mathbf{k}), \qquad v_r^\dagger(\mathbf{k})v_s(\mathbf{k}) = \frac{iE}{m}\bar{v}_r(\mathbf{k})v_s(\mathbf{k})$$

The idea is to use the Dirac equation twice, once on $u^\dagger(\mathbf{k})$ (or $v^\dagger(\mathbf{k})$), and once on $u(\mathbf{k})$ (or $v(\mathbf{k})$). The Dirac equation, in all its forms, is given by

$$\begin{aligned} mu_s(\mathbf{k}) &= i\not{k}u_s(\mathbf{k}), & \bar{u}_s(\mathbf{k})m &= \bar{u}_s(\mathbf{k})i\not{k} \\ mv_s(\mathbf{k}) &= -i\not{k}v_s(\mathbf{k}), & \bar{v}_s(\mathbf{k})m &= -\bar{v}_s(\mathbf{k})i\not{k} \end{aligned}$$

We thus compute:

$$\begin{aligned} u_r^\dagger(\mathbf{k})u_s(\mathbf{k}) &= -\frac{1}{2}\left(\bar{u}_r(\mathbf{k})\gamma^0 u_s(\mathbf{k}) + \bar{u}_r(\mathbf{k})\gamma^0 u_s(\mathbf{k})\right) = -\frac{i}{2m}\left(\bar{u}_r(\mathbf{k})\not{k}\gamma^0 u_s(\mathbf{k}) + \bar{u}_r(\mathbf{k})\gamma^0\not{k}u_s(\mathbf{k})\right) \\ &= -\frac{i}{2m}\bar{u}_r(\mathbf{k})k_\mu\{\gamma^\mu,\gamma^0\}u_s(\mathbf{k}) = \frac{ik_0}{m}\bar{u}_r(\mathbf{k})u_s(\mathbf{k}) = -\frac{iE}{m}\bar{u}_r(\mathbf{k})u_s(\mathbf{k}) = 2E\delta_{rs} \\ v_r^\dagger(\mathbf{k})v_s(\mathbf{k}) &= +\frac{1}{2}\left(\bar{v}_r(\mathbf{k})\gamma^0 v_s(\mathbf{k}) + \bar{v}_r(\mathbf{k})\gamma^0 v_s(\mathbf{k})\right) = +\frac{iE}{m}\bar{v}_r(\mathbf{k})v_s(\mathbf{k}) = 2E\delta_{rs} \end{aligned}$$

In the last step of both calculations we use the normalizations

$$\bar{u}_r(\mathbf{k})u_s(\mathbf{k}) = 2im\delta_{rs}, \qquad \bar{v}_r(\mathbf{k})v_s(\mathbf{k}) = -2im\delta_{rs}$$

Question 3: Stress Tensor and Hamiltonian for the Dirac Theory. (21 points)

(a) The Dirac action is translationally invariant. Use the Noether procedure the construct the conserved currents $\Theta^{\mu\nu}$, i.e. the energy-momentum tensor.
The 'charge' density Θ^{00} for time translation is the energy density. Show that Θ^{00} indeed coincides with the Hamiltonian density $\mathcal{H}$ derived in class, i.e.

$$\Theta^{00} = \mathcal{H} = i\bar{\psi}(\gamma^i\partial_i - m)\psi$$

We begin with the Lagrangian density $\mathcal{L} = -i\bar{\psi}(\partial\!\!\!/ - m)\psi$.
The Noether current for a transformation is given by

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_A)}\delta\Phi_A - \mathcal{F}^\mu, \qquad \delta\mathcal{L} = \partial_\mu\mathcal{F}^\mu$$

Under an spacetime transformation $x^\mu \to x^\mu + \epsilon^\mu$ the Lagrangian density transforms as $\delta\mathcal{L} = -\epsilon^\mu\partial_\mu\mathcal{L}$. The fields transform as $\delta\psi = -\epsilon^\mu\partial_\mu\psi$. Putting everything together, we have

$$j^\mu = -i\bar{\psi}\gamma^\mu(-\epsilon^\nu\partial_\nu\psi) + \epsilon^\mu\mathcal{L} = \epsilon^\nu(i\bar{\psi}\gamma^\mu\partial_\nu\psi - i\bar{\psi}\eta_{\nu\mu}(\partial\!\!\!/ - m)\psi) = i\epsilon_\nu\bar{\psi}(\gamma^{\mu\nu} - \eta^{\mu\nu}(\partial\!\!\!/ - m))\psi$$

We find that $j^\mu = \epsilon_\nu\Theta^{\mu\nu}$ for the stress-energy tensor

$$\Theta^{\mu\nu} = i\bar{\psi}(\gamma^\mu\partial^\nu - \eta^{\mu\nu}(\partial\!\!\!/ - m))\psi$$

In particular, the energy density is

$$\Theta^{00} = i\bar{\psi}(\gamma^0\partial^t + (\partial\!\!\!/ - m))\psi = i\bar{\psi}(-\gamma^0\partial_t + \gamma^0\partial_t + \gamma^i\partial_i - m)\psi = i\bar{\psi}(\gamma^i\partial_i - m)\psi = \mathcal{H}$$

(b) Show that using the Dirac equation, the Hamiltonian can be written as

$$H = i\int d^3x\psi^\dagger\partial_t\psi$$

Express H in terms of the operators $a^s_{\mathbf{k}}$, $a^{s\dagger}_{\mathbf{k}}$, $c^s_{\mathbf{k}}$, $c^{s\dagger}_{\mathbf{k}}$.

The Dirac equation tells us that

$$0 = i\bar{\psi}(\partial\!\!\!/ - m)\psi = i\bar{\psi}(\gamma^0\partial_0 + \gamma^i\partial_i - m)\psi \qquad \Longrightarrow \qquad i\bar{\psi}(\gamma^i\partial_i - m)\psi = -i\bar{\psi}\gamma^0\partial_0\psi = i\psi^\dagger\partial_t\psi$$

Therefore,

$$H = \int d^3x\Theta^{00} = \int d^3x i\bar{\psi}(\gamma^i\partial_i - m)\psi = i\int d^3x\psi^\dagger\partial_t\psi$$

Now we substitute the mode expansion:

$$\psi(x) = \int \frac{d\!\!\!^{-}{}^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\left[a^s_{\mathbf{k}}u_s(\mathbf{k})e^{ik\cdot x} + c^{s\dagger}_{\mathbf{k}}v_s(\mathbf{k})e^{-ik\cdot x}\right]$$

In terms of creation and annihilation operators, the Hamiltonian is thus

$$\begin{aligned}
H &= i\int d^3\mathbf{x}\frac{d\!\!\!^{-}{}^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\frac{d\!\!\!^{-}{}^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\left[a^{r\dagger}_{\mathbf{k}}u^\dagger_r(\mathbf{k})e^{-ik\cdot x} + c^r_{\mathbf{k}}v^\dagger_r(\mathbf{k})e^{ik\cdot x}\right](-ik'_0)\left[a^s_{\mathbf{k}'}u_s(\mathbf{k}')e^{ik'\cdot x} - c^{s\dagger}_{\mathbf{k}'}v_s(\mathbf{k}')e^{-ik'\cdot x}\right] \\
&= \int d^3\mathbf{x}\frac{d\!\!\!^{-}{}^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\frac{d\!\!\!^{-}{}^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}k'_0\Big[a^{r\dagger}_{\mathbf{k}}a^s_{\mathbf{k}'}u^\dagger_r(\mathbf{k})u_s(\mathbf{k}')e^{-i(k-k')\cdot x} - a^{r\dagger}_{\mathbf{k}}c^{s\dagger}_{\mathbf{k}'}u^\dagger_r(\mathbf{k})v_s(\mathbf{k}')e^{-i(k+k')\cdot x} \\
&\qquad\qquad + c^r_{\mathbf{k}}a^s_{\mathbf{k}'}v^\dagger_r(\mathbf{k})u_s(\mathbf{k}')e^{i(k+k')\cdot x} - c^r_{\mathbf{k}}c^{s\dagger}_{\mathbf{k}'}v^\dagger_r(\mathbf{k})v_s(\mathbf{k}')e^{i(k-k')\cdot x}\Big] \\
&= \int\frac{d\!\!\!^{-}{}^3\mathbf{k}}{2}\left[a^{r\dagger}_{\mathbf{k}}a^s_{\mathbf{k}}u^\dagger_r(\mathbf{k})u_s(\mathbf{k}) - e^{2i\omega_{\mathbf{k}}t}a^{r\dagger}_{\mathbf{k}}c^{s\dagger}_{-\mathbf{k}}u^\dagger_r(\mathbf{k})v_s(-\mathbf{k}) + e^{-2i\omega_{\mathbf{k}}t}c^r_{\mathbf{k}}a^s_{-\mathbf{k}}v^\dagger_r(\mathbf{k})u_s(-\mathbf{k}) - c^r_{\mathbf{k}}c^{s\dagger}_{\mathbf{k}}v^\dagger_r(\mathbf{k})v_s(\mathbf{k})\right] \\
&= \frac{1}{2}\int d\!\!\!^{-}{}^3\mathbf{k}\left[a^{r\dagger}_{\mathbf{k}}a^s_{\mathbf{k}}u^\dagger_r(\mathbf{k})u_s(\mathbf{k}) - c^r_{\mathbf{k}}c^{s\dagger}_{\mathbf{k}}v^\dagger_r(\mathbf{k})v_s(\mathbf{k})\right] \\
&= \int d\!\!\!^{-}{}^3\mathbf{k}\ \omega_{\mathbf{k}}\left[a^{s\dagger}_{\mathbf{k}}a^s_{\mathbf{k}} + c^{s\dagger}_{\mathbf{k}}c^s_{\mathbf{k}} - 2(2\pi)^3\delta^{(3)}(0)\right] = \int d\!\!\!^{-}{}^3\mathbf{k}\omega_{\mathbf{k}}(N_{\mathbf{k}} + \bar{N}_{\mathbf{k}}) + E_0
\end{aligned}$$

In the third equality we integrate over x to get a delta function in $\mathbf{k}$ and $\mathbf{k}'$, and subsequently do the integral over $\mathbf{k}'$. In the fourth equality we use the identities $u_r^\dagger(\mathbf{k})v_s(-\mathbf{k}) = v_r^\dagger(\mathbf{k})u_s(-\mathbf{k}) = 0$. In the fifth equality, we use $\{c_{\mathbf{k}}^r, c_{\mathbf{k}'}^{s\dagger}\} = \delta_{rs}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')$ to move the c-creation operator to the left, and the normalization $u_r^\dagger(\mathbf{k})u_s(\mathbf{k}) = v_r^\dagger(\mathbf{k})v_s(\mathbf{k}) = 2k^0\delta_{rs}$. In the final equality, we identify

$$N_{\mathbf{k}} = a_{\mathbf{k}}^{s\dagger}a_{\mathbf{k}}^s, \qquad \bar{N}_{\mathbf{k}} = c_{\mathbf{k}}^{s\dagger}c_{\mathbf{k}}^s, \qquad E_0 = -\int d^3\mathbf{k}\ 2\omega_{\mathbf{k}}\delta^{(3)}(0)$$

(c) What is the vacuum energy density? Discuss the differences with that of a scalar.

The vacuum energy E_0^ψ is identified in (b). To get the vacuum energy density ε_0^ψ, we use that $\delta^{(0)} = \int d^3\mathbf{x}1$ and take the integrand:

$$\varepsilon_0^\psi = -\int d^3\mathbf{k}\ 2\omega_{\mathbf{k}} = -4\int d^3\mathbf{k}\frac{\omega_{\mathbf{k}}}{2}$$

We can compare this to a complex scalar field:

$$\varepsilon_0^\phi = 2\int d^3\mathbf{k}\frac{\omega_{\mathbf{k}}}{2}$$

A real scalar field has 2 degrees of freedom per momentum $\mathbf{k}$, each of which is a harmonic oscillator with negative vacuum energy $\omega_{\mathbf{k}}/2$. A Dirac field has 4 degrees of freedom per momentum $\mathbf{k}$ ($u_{1,2}$, $v_{1,2}$), each of which is a harmonic oscillator with positive vacuum energy $-\omega_{\mathbf{k}}/2$. In particular, if we have 2 complex scalars for each Dirac fermion, then the vacuum energy is exactly zero.

Question 4: Angular Momentum Operators (30 points)

The Dirac action is Lorentz invariant.

(a) Write down an infinitesimal Lorentz transformation for ψ.

We consider an infinitesimal Lorentz transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu$.

A spinor and its conjugate transform as

$$
\begin{aligned}
\delta\psi_\alpha(x) &= \psi'_\alpha(x) - \psi_\alpha(x) = S_\alpha{}^\beta(\Lambda)\psi_\beta(\Lambda^{-1}x) - \psi_\alpha(x) \\
&= \left(\delta_\alpha^\beta - \frac{i}{2}\omega_{\rho\sigma}(\Sigma^{\rho\sigma})_\alpha{}^\beta\right)\left(\psi_\beta(x) + \frac{i}{2}\omega_{\kappa\lambda}i(\eta^{\kappa\mu}\delta_\nu^\lambda - \eta^{\lambda\mu}\delta_\nu^\kappa)x_\nu\partial_\mu\psi_\beta(x)\right) - \psi_\alpha(x) \\
&= -\omega_{\rho\sigma}\left(\frac{i}{2}(\Sigma^{\rho\sigma})_\alpha{}^\beta + \delta_\alpha^\beta x^\sigma\partial^\rho\right)\psi_\beta(x) \\
\delta\bar\psi_\alpha(x) &= \bar\psi'_\alpha(x) - \bar\psi_\alpha(x) = \bar\psi_\beta(\Lambda^{-1}x)(S^{-1}(\Lambda))^\beta{}_\alpha - \bar\psi_\alpha(x) \\
&= \left(\bar\psi_\beta(x) + \frac{i}{2}\omega_{\kappa\lambda}i(\eta^{\kappa\mu}\delta_\nu^\lambda - \eta^{\lambda\mu}\delta_\nu^\kappa)\bar\psi_\beta(x)\overleftarrow{\partial_\mu}x_\nu\right)\left(\delta_\alpha^\beta + \frac{i}{2}\omega_{\rho\sigma}(\Sigma^{\rho\sigma})^\beta{}_\alpha\right) - \bar\psi_\alpha(x) \\
&= -\omega_{\rho\sigma}\bar\psi_\beta(x)\left(-\frac{i}{2}(\Sigma^{\rho\sigma})^\beta{}_\alpha + \delta_\alpha^\beta\overleftarrow{\partial^\rho}x^\sigma\right)
\end{aligned}
$$

(b) Use the Noether procedure to construct the conserved charges $M^{\mu\nu}$ for with Lorentz transformations, and show that $M^{\mu\nu}$ can be written in terms of a 'spin' part and 'orbital angular momentum' part,

$$M^{\mu\nu} = S^{\mu\nu} + L^{\mu\nu}$$

Show that the orbital part $L^{\mu\nu}$ has the same form as that of a scalar

$$L^{\mu\nu} = \int d^3x(x^\mu\Theta^{0\nu} - x^\nu\Theta^{0\mu})$$

Where $\Theta^{\mu\nu}$ is the energy-momentum tensor from 3(a). Show that the spin part $S^{\mu\nu}$ can be written as

$$S^{\mu\nu} = -\int d^3x\psi^\dagger\Sigma^{\mu\nu}\psi$$

We start with $\mathcal{L} = -i\bar\psi(\partial\!\!\!/ - m)\psi$. Under the Lorentz transform in (a), $\mathcal{L}$ transforms as

$$
\begin{aligned}
\delta\mathcal{L} &= -i(\delta\bar\psi(\partial\!\!\!/ - m)\psi + \bar\psi(\partial\!\!\!/ - m)\delta\psi) \\
&= i\omega_{\rho\sigma}\left[\bar\psi\left(-\frac{i}{2}\Sigma^{\rho\sigma} + \overleftarrow{\partial^\rho}x^\sigma\right)(\partial\!\!\!/ - m)\psi + \bar\psi(\partial\!\!\!/ - m)\left(\frac{i}{2}\Sigma^{\rho\sigma} + x^\sigma\partial^\rho\right)\psi\right] \\
&= -\omega_{\rho\sigma}x^\sigma\partial^\rho\mathcal{L} + i\omega_{\rho\sigma}\bar\psi\gamma^\sigma\partial^\rho\psi + \frac{1}{2}\omega_{\rho\sigma}\bar\psi(\Sigma^{\rho\sigma}\partial\!\!\!/ - \partial\!\!\!/\Sigma^{\rho\sigma})\psi \\
&= -\omega_{\rho\sigma}x^\sigma\partial^\rho\mathcal{L} + i\omega_{\rho\sigma}\bar\psi\gamma^\sigma\partial^\rho\psi - \frac{1}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\kappa{}_\lambda\bar\psi\gamma^\lambda\partial_\kappa\psi \\
&= -\omega_{\rho\sigma}x^\sigma\partial^\rho\mathcal{L} = -\omega_{\rho\sigma}\partial^\rho(x^\sigma\mathcal{L})
\end{aligned}
$$

Note that in the third line, the second term comes from the product rule, in particular by $\partial\!\!\!/$ acting on the x^σ of the previous expression. In the fourth line we use that $[\Sigma^{\rho\sigma}, \gamma^\mu] = -(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu$. In the last line we use the explicit form of $(\mathcal{J}^{\rho\sigma})^\mu{}_\nu$, and that $\omega_{\rho\sigma}$ is antisymmetric.

Therefore, the Noether current is:

$$\begin{aligned} j^\rho &= \frac{\partial \mathcal{L}}{\partial(\partial_\rho \Phi_A)}\delta\Phi_A - \mathcal{F}^\rho = -i\bar{\psi}\gamma^\rho \delta\psi + \omega^{\rho\sigma}x_\sigma \mathcal{L} \\ &= \omega_{\mu\nu}\left(-\frac{1}{2}\bar{\psi}\gamma^\rho \Sigma^{\mu\nu}\psi + ix^\nu \bar{\psi}\gamma^\rho \partial^\mu \psi + \eta^{\mu\rho}x^\nu \mathcal{L}\right) \\ &= \omega_{\mu\nu}\left(-\frac{1}{2}\bar{\psi}\gamma^\rho \Sigma^{\mu\nu}\psi + \Theta^{\rho\mu}x^\nu\right) \end{aligned}$$

where we recall the energy-momentum tensor

$$\Theta^{\mu\nu} = i\bar{\psi}(\gamma^\mu \partial^\nu - \eta^{\mu\nu}(\not{\partial} - m))\psi = i\bar{\psi}\gamma^\mu \partial^\nu \psi + \eta^{\mu\nu}\mathcal{L}$$

We have an independent Noether current for each independent component of $\omega_{\mu\nu}$, of which there are 6. Therefore, we can define $j^\rho = -\frac{1}{2}\omega_{\mu\nu}J^{\rho\mu\nu}$, for the conserved currents

$$J^{\rho\mu\nu} = \frac{1}{2}\bar{\psi}\gamma^\rho \Sigma^{[\mu\nu]}\psi - \Theta^{\rho[\mu}x^{\nu]} = \bar{\psi}\gamma^\rho \Sigma^{\mu\nu}\psi + x^\mu \Theta^{\rho\nu} - x^\nu \Theta^{\rho\mu}$$

The Noether charges are

$$M^{\mu\nu} = \int d^3x J^{0\mu\nu} = \int d^3x \left[-\psi^\dagger \Sigma^{\mu\nu}\psi + x^\mu \Theta^{0\nu} - x^\nu \Theta^{0\mu}\right] = S^{\mu\nu} + L^{\mu\nu}$$

where we identify

$$S^{\mu\nu} = -\int d^3x \psi^\dagger \Sigma^{\mu\nu}\psi, \qquad L^{\mu\nu} = \int d^3x (x^\mu \Theta^{0\nu} - x^\nu \Theta^{0\mu})$$

Note that $S^{\mu\nu}$ and $L^{\mu\nu}$ are not separately conserved.

(c) Express the $S^{\mu\nu}$ in terms of the operators $a^s_{\mathbf{k}}$, $a^{s\dagger}_{\mathbf{k}}$, $c^s_{\mathbf{k}}$, $c^{s\dagger}_{\mathbf{k}}$.
We need the mode expansion

$$\psi(x) = \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\left[a^s_{\mathbf{k}}u_s(\mathbf{k})e^{ik\cdot x} + c^{s\dagger}_{\mathbf{k}}v_s(\mathbf{k})e^{-ik\cdot x}\right]$$

The process is straightforwards, very similar to 2(b). We substitute the mode expansion into $S^{\mu\nu}$, and expand to get 4 terms. We perform the integral over $\mathbf{x}$ to get a $\delta^{(3)}(\mathbf{k}-\mathbf{k}')$ or $\delta^{(3)}(\mathbf{k}+\mathbf{k}')$ for each term, and perform the $\mathbf{k}'$ integral to set $\mathbf{k}' = \mathbf{k}$ for the $u^\dagger \Sigma^{\mu\nu}u$ and $v^\dagger \Sigma^{\mu\nu}v$ terms, and set $\mathbf{k}' = -\mathbf{k}$ for the $u^\dagger \Sigma^{\mu\nu}v$ and $v^\dagger \Sigma^{\mu\nu}u$ terms. The result is

$$\begin{aligned} S^{\mu\nu} = -\int \frac{đ^3\mathbf{k}}{2\omega_{\mathbf{k}}}\Big[& a^{r\dagger}_{\mathbf{k}}a^s_{\mathbf{k}}u^\dagger_r(\mathbf{k})\Sigma^{\mu\nu}u_s(\mathbf{k}) + e^{2i\omega_{\mathbf{k}}t}a^{r\dagger}_{\mathbf{k}}c^{s\dagger}_{-\mathbf{k}}u^\dagger_r(\mathbf{k})\Sigma^{\mu\nu}v_s(-\mathbf{k}) \\ & + e^{-2i\omega_{\mathbf{k}}t}c^r_{\mathbf{k}}a^s_{-\mathbf{k}}v^\dagger_r(\mathbf{k})\Sigma^{\mu\nu}u_s(-\mathbf{k}) + c^r_{\mathbf{k}}c^{s\dagger}_{\mathbf{k}}v^\dagger_r(\mathbf{k})\Sigma^{\mu\nu}v_s(\mathbf{k})\Big] \end{aligned}$$

(d) From the expression in (c) for S^{ij}, keep only the time-independent part, denoted $\tilde{S}^{ij}$. Define

$$\mathbf{J}^2 = \frac{1}{2}\tilde{S}^{ij}\tilde{S}_{ij}$$

Show that the one-particle states constructed by acting $a^{s\dagger}_{\mathbf{k}}$ and $c^{s\dagger}_{\mathbf{k}}$ with $\mathbf{k} = 0$ (i.e. in the rest frame) on the vacuum are eigenstates of $\mathbf{J}^2$ with eigenvalues corresponding to that of a spin-$\frac{1}{2}$ particle.

We note that $S^{\mu\nu}$ has time-dependent terms. This is fine, since only the combination $M^{\mu\nu} = S^{\mu\nu} + L^{\mu\nu}$ is expected to be conserved. Thus in the analysis of angular momentum we may throw away these terms. We further normal order our operator by placing annihilation operators to the right, to get $\tilde{S}^{\mu\nu}$. The spatial components of this tensor are

$$\tilde{S}^{ij} = -\int \frac{đ^3\mathbf{k}}{2\omega_{\mathbf{k}}}\left[a^{r\dagger}_{\mathbf{k}}a^{s}_{\mathbf{k}}u^\dagger_r(\mathbf{k})\Sigma^{\mu\nu}u_s(\mathbf{k}) - c^{s\dagger}_{\mathbf{k}}c^{r}_{\mathbf{k}}v^\dagger_r(\mathbf{k})\Sigma^{\mu\nu}v_s(\mathbf{k})\right]$$

Note that the second term picks up a -1 from normal ordering, because the operators are fermionic.

Now we compute the action of $\mathbf{J}^2 = \frac{1}{2}\tilde{S}^{ij}\tilde{S}_{ij}$ on the 1-particle state $a^{t\dagger}_{\mathbf{0}}|0\rangle$.

$$\begin{aligned}
\mathbf{J}^2 a^{t\dagger}_{\mathbf{0}}|0\rangle &= \frac{1}{2}\int \frac{đ^3\mathbf{k}}{2\omega_{\mathbf{k}}}\frac{đ^3\mathbf{k}'}{2\omega_{\mathbf{k}'}}\left[u^\dagger_r(\mathbf{k})\Sigma^{ij}u_s(\mathbf{k})\right]\left[u^\dagger_{r'}(\mathbf{k}')\Sigma_{ij}u_{s'}(\mathbf{k}')\right]a^{r\dagger}_{\mathbf{k}}a^{s}_{\mathbf{k}}a^{r'\dagger}_{\mathbf{k}'}a^{s'}_{\mathbf{k}'}a^{t\dagger}_{\mathbf{0}}|0\rangle \\
&= \frac{1}{2}\frac{1}{(2\omega_{\mathbf{0}})^2}u^\dagger_r(\mathbf{0})\Sigma^{ij}u_s(\mathbf{0})u^\dagger_s(\mathbf{0})\Sigma_{ij}u_t(\mathbf{0})a^{r\dagger}_{\mathbf{0}}|0\rangle \\
&= \frac{1}{8m^2}u^\dagger_r(\mathbf{0})\Sigma^{ij}\Sigma_{ij}[u_s(\mathbf{0})u^\dagger_s(\mathbf{0})]u_t(\mathbf{0})a^{r\dagger}_{\mathbf{0}}|0\rangle \\
&= \frac{1}{8m^2}u^\dagger_r(\mathbf{0})\Sigma^{ij}\Sigma_{ij}u_s(\mathbf{0})(2m\delta_{st})a^{r\dagger}_{\mathbf{0}}|0\rangle = \frac{1}{4m}u^\dagger_r(\mathbf{0})\Sigma^{ij}\Sigma_{ij}u_t(\mathbf{0})a^{r\dagger}_{\mathbf{0}}|0\rangle \\
&= \frac{1}{4m}\frac{3}{2}u^\dagger_r(\mathbf{0})u_t(\mathbf{0})a^{r\dagger}_{\mathbf{0}}|0\rangle = \frac{3}{4}a^{t\dagger}_{\mathbf{0}}|0\rangle
\end{aligned}$$

In the first line, note that any terms containing c's vanish, since the lowering operator anticommutes with $a^{t\dagger}$ to annihilate the vacuum. In the second line we use that

$$a^{r\dagger}_{\mathbf{k}}a^{s}_{\mathbf{k}}a^{r'\dagger}_{\mathbf{k}'}a^{s'}_{\mathbf{k}'}a^{t\dagger}_{\mathbf{0}}|0\rangle = \delta^{s't}(2\pi)^3\delta^{(3)}(\mathbf{k}')a^{r\dagger}_{\mathbf{k}}a^{s}_{\mathbf{k}}a^{r'\dagger}_{\mathbf{k}'}|0\rangle = \delta^{s't}\delta^{sr'}(2\pi)^6\delta^{(3)}(\mathbf{k}')\delta^{(3)}(\mathbf{k}-\mathbf{k}')a^{r\dagger}_{\mathbf{k}}|0\rangle$$

In the third line we use that $u_s(\mathbf{0})u^\dagger_s(\mathbf{0}) = i(i\not{k} + m)|_{\mathbf{k}=0} = m(i + \gamma^0)$. Further noting that $\{\gamma^0, \gamma^i\} = 0$, we see that this commutes with $\Sigma^{ij} = \frac{i}{4}[\gamma^i, \gamma^j]$. In the fourth line (and fifth line), we use the identity $u^\dagger_r(\mathbf{k})u_s(\mathbf{k}) = 2E\delta_{rs}$. Finally, in the fifth line we also use that

$$\begin{aligned}
\Sigma^{ij}\Sigma_{ij} &= -\frac{1}{16}[\gamma^i, \gamma^j][\gamma^i, \gamma^j] = -\frac{1}{8}(\gamma^i\gamma^j\gamma^i\gamma^j - \gamma^i\gamma^j\gamma^j\gamma^i) \\
&= -\frac{1}{8}(-(\gamma^i\gamma^i)^2 + 2\gamma^i\gamma^i - 3\gamma^i\gamma^i) = -\frac{1}{8}(-9 + 6 - 9) = \frac{3}{2}
\end{aligned}$$

Therefore, we see that the one-particle state $a^{t\dagger}_{\mathbf{0}}|0\rangle$ is an eigenstate of $\mathbf{J}^2$ with eigenvalue $s(s+1) = \frac{3}{4}$, as expected from a spin-1/2 particle.

The calculation is almost identical with the one-particle state $c^{t\dagger}_{\mathbf{0}}|0\rangle$: one merely replaces a's with c's and u's with v's. The one difference is in the third line, where instead we need to use the identity $v_s(\mathbf{0})v^\dagger_s(\mathbf{0}) = i(i\not{k} - m)|_{\mathbf{k}=0} = m(-i + \gamma^0)$. However, this still commutes with Σ^{ij}, so the same proof follows through.

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Spring 2023