8.323 Problem Set 9 Solutions

April 18, 2023

Question 1: Some Identities (10 points)
Define γ^5 as

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

Show it has the following properties: (a) $(\gamma^5)^\dagger = \gamma^5$ and $(\gamma^5)^2 = 1$
We compute:

$$\begin{aligned}(\gamma^5)^\dagger &= -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger = -i(\gamma^0\gamma^3\gamma^0)(\gamma^0\gamma^2\gamma^0)(\gamma^0\gamma^1\gamma^0)(\gamma^0\gamma^0\gamma^0)\\ &= -(-1)^4 i\gamma^0\gamma^3\gamma^2\gamma^1 = -i(-1)^3\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5\end{aligned}$$

In the 2nd equality we use that $(\gamma^0)^2 = -1$. In the 4th equality we use the anticommutation properties of the gamma matrices.

Furthermore,

$$(\gamma^5)^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -(-1)^3(-1)^2(-1)^1\gamma^0\gamma^0\gamma^1\gamma^1\gamma^2\gamma^2\gamma^3\gamma^3 = -\eta^{00}\eta^{11}\eta^{22}\eta^{33} = -1$$

In the 3rd equality we use $(\gamma^\mu)^2 = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}$ (no sum over μ).

(b) $\{\gamma^5, \gamma^\mu\} = 0$ and $\mathrm{Tr}\gamma^5 = 0$
This is not difficult to compute for each $\mu = 0, 1, 2, 3$. Using the anticommutation relations, we have:

$$\begin{aligned}\{\gamma^5, \gamma^0\} &= i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 + \gamma^0\gamma^0\gamma^1\gamma^2\gamma^3) = i((-1)^3\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 + \gamma^0\gamma^0\gamma^1\gamma^2\gamma^3) = 0\\ \{\gamma^5, \gamma^1\} &= i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1 + \gamma^1\gamma^0\gamma^1\gamma^2\gamma^3) = i((-1)^2\gamma^0\gamma^1\gamma^1\gamma^2\gamma^3 + (-1)\gamma^0\gamma^1\gamma^1\gamma^2\gamma^3) = 0\\ \{\gamma^5, \gamma^2\} &= i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^2 + \gamma^2\gamma^0\gamma^1\gamma^2\gamma^3) = i((-1)\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3 + (-1)^2\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3) = 0\\ \{\gamma^5, \gamma^3\} &= i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3 + \gamma^3\gamma^0\gamma^1\gamma^2\gamma^3) = i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3 + (-1)^3\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3) = 0\end{aligned}$$

Furthermore,

$$\begin{aligned}\mathrm{Tr}\gamma^5 &= \mathrm{Tr}(\gamma^5\gamma^0\gamma_0) = \frac{1}{2}(\mathrm{Tr}(\gamma^5\gamma^0\gamma_0) + \mathrm{Tr}(\gamma_0\gamma^0\gamma^5))\\ &= \frac{1}{2}(\mathrm{Tr}(\gamma^5\gamma^0\gamma_0) + \mathrm{Tr}(\gamma^0\gamma^5\gamma_0)) = \frac{1}{2}\mathrm{Tr}(\{\gamma^5, \gamma^0\}\gamma_0\} = 0\end{aligned}$$

In the 3rd equality we use the cyclicity of the trace, and in the last equality we use the previous result that $\{\gamma^5, \gamma^0\} = 0$. Note that instead of γ^0 we could have chosen any γ^μ matrix.

Question 2: Feynman Propagator for Dirac Spinors (10 points)
Show that the Feynman Green function

$$D_F^{\alpha\beta}(x-y) = \langle 0|\mathrm{T}\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle = i(\not{\partial} + m)_{\alpha\beta} G_F(x-y)$$

where G_F is the Feynman propagator for a free complex scalar of the same mass m.
We start by computing the Wightman function using the mode expansion of ψ.

$$\begin{aligned}
\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle &= \int \frac{\not{d}^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\frac{\not{d}^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\langle 0|a^r_\mathbf{k} a^{\dagger s}_{\mathbf{k}'}|0\rangle u^r_\alpha(\mathbf{k})\bar{u}^s_\beta(\mathbf{k}')e^{i(k\cdot x - k'\cdot y)} \\
&= \int \frac{\not{d}^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\frac{\not{d}^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\delta_{rs}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')u^r_\alpha(\mathbf{k})\bar{u}^s_\beta(\mathbf{k}')e^{i(k\cdot x - k'\cdot y)} = \int \frac{\not{d}^3\mathbf{k}}{2\omega_\mathbf{k}}\Lambda^+_{\alpha\beta}(\mathbf{k})e^{ik\cdot(x-y)} \\
&= i(\not{\partial} + m)_{\alpha\beta}\int \frac{\not{d}^3\mathbf{k}}{2\omega_\mathbf{k}}e^{ik\cdot(x-y)} = i(\not{\partial}+m)_{\alpha\beta}G_+(x-y)
\end{aligned}$$

In the 3nd equality we use the identity

$$\sum_s u^s_\alpha(\mathbf{k})\bar{u}^s_\beta(\mathbf{k}) = \Lambda^+_{\alpha\beta}(\mathbf{k}) = i(i\not{k} + m)_{\alpha\beta}$$

In the last equality we identify $G_+(x-y) = \int \frac{\not{d}^3\mathbf{k}}{2\omega_\mathbf{k}}e^{ik\cdot(x-y)}$.

Similarly, we have:

$$\begin{aligned}
\langle 0|\bar{\psi}_\beta(y)\psi_\alpha(x)|0\rangle &= \int \frac{\not{d}^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\frac{\not{d}^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\langle 0|c^s_\mathbf{k} c^{\dagger r}_{\mathbf{k}'}|0\rangle v^r_\alpha(\mathbf{k})\bar{v}^s_\beta(\mathbf{k}')e^{-i(k\cdot x - k'\cdot y)} \\
&= \int \frac{\not{d}^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\frac{\not{d}^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\delta_{rs}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')v^r_\alpha(\mathbf{k})\bar{v}^s_\beta(\mathbf{k}')e^{-i(k\cdot x - k'\cdot y)} = \int \frac{\not{d}^3\mathbf{k}}{2\omega_\mathbf{k}}\Lambda^-_{\alpha\beta}(\mathbf{k})e^{-ik\cdot(x-y)} \\
&= -i(\not{\partial} + m)_{\alpha\beta}\int \frac{\not{d}^3\mathbf{k}}{2\omega_\mathbf{k}}e^{-ik\cdot(x-y)} = -i(\not{\partial}+m)_{\alpha\beta}G_+(y-x)
\end{aligned}$$

Now we compute the Feynman Green's function:

$$\begin{aligned}
D_F^{\alpha\beta}(x-y) &= \theta(x^0-y^0)\langle 0|\psi^\alpha(x)\bar{\psi}^\beta(y)|0\rangle - \theta(y^0-x^0)\langle 0|\bar{\psi}^\beta(y)\psi^\alpha(x)|0\rangle \\
&= \theta(x^0-y^0)i(\not{\partial}_x+m)^{\alpha\beta}G_+(x-y) + \theta(y^0-x^0)i(\not{\partial}_x+m)^{\alpha\beta}G_+(y-x) \\
&= i(\not{\partial}_x+m)^{\alpha\beta}G_F(x-y) - i(\not{\partial}_x\theta(x^0-y^0))G_+(x-y) - i(\not{\partial}_x\theta(y^0-x^0))G_+(y-x) \\
&= i(\not{\partial}_x+m)^{\alpha\beta}G_F(x-y) - i\gamma^0(\delta(x^0-y^0)G_+(x-y) - \delta(x^0-y^0)G_+(y-x)) \\
&= i(\not{\partial}_x+m)^{\alpha\beta}G_F(x-y)
\end{aligned}$$

In the second last line we use that $\partial_{x^0}\theta(x^0-y^0) = -\partial_{x^0}\theta(y^0-x^0) = \delta(x^0-y^0)$. In the last line, we use that $G_+(0,\mathbf{x}-\mathbf{y}) = G_+(0,\mathbf{y}-\mathbf{x})$, which can easily be seen from the formula for G_+ above. This leaves us with the resired result.

Question 3: Chiral and Majorana Fermions. (50 points)

We consider the chiral representation, and write a Dirac spinor in terms of 2 chiral spinors ψ_L, ψ_R as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

(a) Show that under a rotation with parameters $\omega_{ij} = \epsilon_{ijk}\theta_k$, $\psi_{L,R}$ transform as

$$\psi'_L(x') = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}\psi_L(x), \qquad \psi'_R(x') = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}\psi_R(x)$$

A Dirac fermion transforms as

$$\psi'(x') = S(\Lambda)\psi(x) = e^{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}\psi(x)$$

In the chiral basis,

$$\Sigma^{ij} = \frac{i}{4}\left[\begin{pmatrix} 0 & i\bar{\sigma}^i \\ i\sigma^i & 0 \end{pmatrix}, \begin{pmatrix} 0 & i\bar{\sigma}^j \\ i\sigma^j & 0 \end{pmatrix}\right] = \frac{i}{4}\begin{pmatrix} [\sigma^i, \sigma^j] & 0 \\ 0 & [\sigma^i, \sigma^j] \end{pmatrix}$$

Therefore for a rotation,

$$\begin{aligned} \psi'_L(x') &= e^{\frac{1}{8}\omega_{ij}[\sigma^i,\sigma^j]}\psi_L(x) = e^{\frac{1}{8}\epsilon_{ijk}\theta_k(2i\epsilon_{ijl}\sigma_l)}\psi_L(x) = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}\psi_L(x) \\ \psi'_R(x') &= e^{\frac{1}{8}\omega_{ij}[\sigma^i,\sigma^j]}\psi_R(x) \qquad\qquad\qquad\qquad = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}\psi_R(x) \end{aligned}$$

(b) Show that under a boost with parameters $\omega_{0i} = \beta_i$, $\psi_{L,R}$ transform as

$$\psi'_L(x') = e^{-\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}}\psi_L(x), \qquad \psi'_R(x') = e^{+\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}}\psi_R(x)$$

Proceeding analogously, in the chiral basis,

$$\Sigma^{0i} = \frac{i}{4}\left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & i\bar{\sigma}^i \\ i\sigma^i & 0 \end{pmatrix}\right] = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} = -\Sigma^{i0}$$

Therefore for a boost,

$$\begin{aligned} \psi'_L(x') &= e^{-\frac{1}{4}(\omega_{0i}\sigma^i - \omega_{i0}\sigma^i)}\psi_L(x) = e^{-\frac{1}{2}\beta_i\sigma^i}\psi_L(x) = e^{-\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}}\psi_L(x) \\ \psi'_R(x') &= e^{+\frac{1}{4}(\omega_{0i}\sigma^i - \omega_{i0}\sigma^i)}\psi_L(x) = e^{+\frac{1}{2}\beta_i\sigma^i}\psi_R(x) = e^{+\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}}\psi_R(x) \end{aligned}$$

(c) The Lagrangian density fo the Dirac theory contains a mass term of the form

$$\mathcal{L} = \cdots + im\bar{\psi}\psi = \cdots - m(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L)$$

Using the transformations in (a), (b), show that the mass term is Lorentz invariant, while a term of the form $m\psi_L^\dagger\psi_L$ or $m\psi_R^\dagger\psi_R$ is not.

We expand the Lagrangian into chiral spinors, in the chiral representation:

$$\begin{aligned} \mathcal{L} = -i\bar{\psi}(\partial\!\!\!/ - m)\psi &= -i(\psi_L^\dagger, \psi_R^\dagger)\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} -m & i\bar{\sigma}^\mu\partial_\mu \\ i\sigma^\mu\partial_\mu & -m \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \\ &= i\psi_L^\dagger\sigma^\mu\partial_\mu\psi_L + i\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L - m(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L) \end{aligned}$$

Under a rotation, $(\psi_L, \psi_R) \to (e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}\psi_L, e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}\psi_R)$, and $(\psi_L^\dagger, \psi_R^\dagger) \to (\psi_L^\dagger e^{-\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}, \psi_R^\dagger e^{-\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}})$ by Hermiticity. The mass terms are both invariant. Under a boost, $(\psi_L, \psi_R) \to (e^{-\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}}\psi_L, e^{+\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}}\psi_R)$, and $(\psi_L^\dagger, \psi_R^\dagger) \to (\psi_L^\dagger e^{-\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}}, \psi_R^\dagger e^{+\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}})$. The mass terms are again invariant, establishing full Lorentz invariance.
However, under a boost $\psi_L^\dagger\psi_L$ or $\psi_R^\dagger\psi_R$ are not invariant:

$$\psi_L^\dagger\psi_L \to \psi_L^\dagger e^{-\boldsymbol{\beta}\cdot\boldsymbol{\sigma}}\psi_L, \qquad \psi_R^\dagger\psi_R \to \psi_R^\dagger e^{+\boldsymbol{\beta}\cdot\boldsymbol{\sigma}}\psi_R$$

(d) The discussion in (c) may give the impression that it is not possible to write down a mass term with ψ_L or ψ_R alone. This is possible, with some more sophistication. For this purpose, first show that

$$\sigma^2\vec{\sigma}\sigma^2 = -\vec{\sigma}^*$$

From this, show that $\sigma^2\psi_L^*$ transforms under Lorentz transformations in the same was as ψ_R.
We can check directly, either by explicit multiplication or using $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$, that

$$\sigma^2\sigma^1\sigma^2 = -\sigma^1 = -(\sigma^1)^*, \qquad \sigma^2\sigma^3\sigma^2 = -\sigma^3 = -(\sigma^3)^*, \qquad \sigma^2\sigma^2\sigma^2 = \sigma^2 = -(\sigma^2)^*$$

Now consider the transformation properties of $\sigma^2\psi_L^*$.
Under rotations,

$$\sigma^2\psi_L'^*(x') = \sigma^2(e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}})^*\psi_L^*(x) = \sigma^2 e^{-\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}^*}\sigma^2\sigma^2\psi_L^*(x) = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}^*}\sigma^2\psi_L^*(x)$$

In the last equality we make use of the above identity. This can be made more precise by expanding the exponential as a power series, inserting $1 = \sigma^2\sigma^2$ between powers of $\boldsymbol{\sigma}$, using said identity, and resumming the series.
Under boosts,

$$\sigma^2\psi_L'^*(x') = \sigma^2(e^{-\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}})^*\psi_L^*(x) = \sigma^2 e^{-\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}^*}\sigma^2\sigma^2\psi_L^*(x) = e^{+\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}^*}\sigma^2\psi_L^*(x)$$

Therefore, $\sigma^2\psi_L^*$ transforms like ψ_R.

(e) From the observation in (d), construct a mass term using ψ_L alone, which is both Lorentz invariant and real. This is called the Majorana mass term. Show that it is identically zero if ψ_L consists of ordinary functions, while it is non-zero if ψ_L are anticommuting variables.
Consider the mass term $m\psi_L^\dagger\sigma^2\psi_L^*$. This is Lorentz invariant, as we showed in (c) that $\psi_L^\dagger\psi_R$ is invariant, and in (d) that $\sigma^2\psi_L^*$ transforms in the same way as ψ_R. To make it real, we add its Hermitian conjugate, which must also be Lorentz invariant. Therefore, consider

$$\mathcal{L}_{\chi m} = -\frac{1}{2}m\psi_L^\dagger\sigma^2\psi_L^* - \frac{1}{2}(m\psi_L^\dagger\sigma^2\psi_L^*)^* = -\frac{m}{2}(\psi_L^\dagger\sigma^2\psi_L^* + \psi_L^T\sigma^2\psi_L)$$

To see whether this vanishes, we write the second term in components:

$$\psi_L^T\sigma^2\psi_L = \psi_{La}(\sigma^2)^a{}_b\psi_L^b = (-1)^\epsilon\psi_L^b(\sigma^2)^a{}_b\psi_{La} = -(-1)^\epsilon\psi_L^b(\sigma^2)_b{}^a\psi_{La} = -(-1)^\epsilon\psi_L^T\sigma^2\psi_L$$

Here we introduce the notation $(-1)^\epsilon$, where $\epsilon = 0$ for regular functions, and $\epsilon = -1$ for Grassmann functions. When $\epsilon = 0$ we see that the term is the negative of itself, and vanishes. The same holds for its Hermitian conjugate, thus the Lagrangian vanishes identically. If the field is Grassmann, we get no such constraint.

(f) Now write down a Lorentz invariant full action using ψ_L alone, including both kinetic and mass terms. Write down equations of motion.

Consider now the Lagrangian

$$\mathcal{L} = i\psi_L^\dagger \sigma^\mu \partial_\mu \psi_L - \frac{m}{2}(\psi_L^\dagger \sigma^2 \psi_L^* + \psi_L^T \sigma^2 \psi_L)$$

Treating ψ_L and ψ_L^* as independent variables, the equations of motion are

$$\begin{aligned} i\sigma^\mu \partial_\mu \psi_L - m\sigma^2 \psi_L^* &= 0 \\ -i\partial_\mu \psi_L^\dagger \bar{\sigma}^\mu - m\psi_L^T \sigma^2 &= 0 \end{aligned}$$

(g) Does the theory of part (f) have a conserved charge? Argue that such a theory can only describe neutral particles.

The theory has the usual conserved charges corresponding to spacetime and Poincaré symmetry. However, there are no internal symmetries, as the mass term is not invariant under a U(1) rotation $\psi \to e^{i\alpha}\psi$. Therefore, there is no U(1) conserved charge, and particle number is not conserved. This cannot describe charged particles, as it would violate charge conservation.

Question 4: Majorana Fermions (10 points)

In the Majorana representation, γ^μ are real, and thus ψ can be chosen to be real. Such a spinor is called a Majorana spinor. This has important physical consequences. Upon quantization, being real, a Majorana particle should not have an antiparticle (or equivalently, it is its own antiparticle).

We discussed in lecture that the concept of a Majorana spinor can be generalized to any representation of γ. If we can find a matrix B satisfying

$$B\gamma^\mu B^{-1} = (\gamma^\mu)^*$$

the Majorana condition is

$$\psi^* = B\psi$$

(a) In lecture we showed that in the chiral representation we can choose $B = \gamma^2$. Solve the Majorana condition in the chiral representation. Show that in this representation a Majorana spinor ψ can be expressed in terms of ψ_L alone.

For $B = \gamma^2$, the Majorana condition in the chiral representation is

$$B\psi = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_L^* \\ \psi_R^* \end{pmatrix} = \psi^*$$

This gives us $\psi_L^* = -i\sigma^2\psi_R$, and $\psi_R^* = i\sigma^2\psi_L$. Therefore $\psi_R = -i(\sigma^2)^*\psi_L^* = i\sigma^2\psi_L^*$, and we can write ψ in terms of purely ψ_L:

$$\psi = \begin{pmatrix} \psi_L \\ i\sigma^2\psi_L^* \end{pmatrix}$$

(b) Plug in the expression (in terms of ψ_L) for the Majorana spinor ψ from (a) into the Dirac action. Show it reduces to the action in part 3(f).

The Dirac Lagrangian becomes:

$$\begin{aligned}\mathcal{L} = -i\bar{\psi}(\partial\!\!\!/ - m)\psi &= i\psi_L^\dagger\sigma^\mu\partial_\mu\psi_L + i\psi_R^\dagger\bar{\sigma}^\mu\partial_\mu\psi_R - m(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L) \\ &= i\psi_L^\dagger\sigma^\mu\partial_\mu\psi_L + i\psi_L^T\sigma^2\bar{\sigma}^\mu\sigma^2\partial_\mu\psi_L^* - im(\psi_L^\dagger\sigma^2\psi_L^* - \psi_L^T\sigma^2\psi_L)\end{aligned}$$

To write this in a more familiar form we integrate the second term by parts and discarding the boundary contribution.

$$\begin{aligned}i\psi_L^T\sigma^2\bar{\sigma}^\mu\sigma^2\partial_\mu\psi_L^* &= -i(\partial_\mu\psi_L^T)\sigma^2\bar{\sigma}^\mu\sigma^2\psi_L^* = -i((\partial_\mu\psi_L^T)\sigma^2\bar{\sigma}^\mu\sigma^2\psi_L^*)^T = i\psi_L^\dagger(\sigma^2\bar{\sigma}^\mu\sigma^2)^T\partial_\mu\psi_L \\ &= i\psi_L^\dagger(\sigma^2\bar{\sigma}^\mu\sigma^2)^*\partial_\mu\psi_L = i\psi_L^\dagger\sigma^\mu\partial_\mu\psi_L\end{aligned}$$

In the third equality, we pick up a sign when we take the transpose because the fermionic fields anticommute. In the 4th equality we use the Hermiticity of the σ^μ's. In the last equality we use the relation from 3(d) that $(\sigma^2\bar{\sigma}^\mu\sigma^2)^* = \sigma^\mu$.

Therefore, the Lagrangian becomes:

$$\mathcal{L} = 2i\psi_L^\dagger\sigma^\mu\partial_\mu\psi_L - im(\psi_L^\dagger\sigma^2\psi_L^* - \psi_L^T\sigma^2\psi_L)$$

This is identical to the Majorana action from problem 3(f) with the field redefinition $\psi_L \to \sqrt{2}e^{-i\pi/4}\psi_L$.