Recitation 4-5

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March 10, 2023

1 The Path-Integral in Quantum Mechanics

1.1 Introduction and Interpretation

Before talking about QFT, we'll discuss the path-integral in QM, which is no more than a $d = 0 + 1$ dimensional QFT. The path-integral formulation of quantum mechanics generalizes the principle of least action. It takes in an action S from a classical theory, and it gives you a quantum theory. This idea is known as path-integral quantization, and alongside canonical quantization it's one of the ways we can bridge the classical and quantum worlds. By and far, I think the path-integral is the most intuitive.

The path-integral can be broken down into 2 fundamental parts.

1. The sole ingredient we feed the path-integral is an action. This is a functional, meaning for each function it returns a number, $S : x(t) \to \mathbb{R}$. Now consider the object $e^{iS[x(t)]/\hbar}$. This is another functional which maps each path to a (now complex) number. The path-integral formulation tells us this is actually a *probability density*, which associates to each path a *weight*.

2. With this probability density, we can compute probabilities. Here one has the formula

$$\langle x_b, t_b | A | x_a, t_a \rangle = \int_{x(t_a)=x_a}^{x(t_b)=x_b} D[x(t)] A e^{iS/\hbar}, \qquad S = \int_{t_a}^{t_b} d\tau L(q, \dot{q})$$

   In particular we can take $A = \mathbb{1}$ to recover what we call the path-integral.

$$Z := \langle x_b, t_b | x_a, t_a \rangle = \int_{x(t_a)=x_a}^{x(t_b)=x_b} D[x(t)] e^{iS/\hbar}$$

   That is, to calculate the probability of starting with an initial state at t_a and ending up in some final state at t_b, we have to sum every single possible path from x_a to x_b, weighted by our density. One may say that the particle takes 'every path', and the weights of different paths differ by a phase. Note that 'all possible paths' includes paths which are not continuous, or those where the particle travels faster than light. This may be eyebrow raising, but is in principle fine–no observables violate causality.

I'd like to emphasize that the crux of the physics lies in point 1, interpreting the object $e^{iS/\hbar}$ as a probability weight. Once you have this, point 2 and everything else is just math, applying the tools of probability theory to do calculations.

You've all seen this idea before, namely in statistical mechanics. In the canonical ensemble one has the Boltzmann weight $e^{-\beta\mathcal{H}}$, and the partition function is akin to the path integral. Again, we have is a probability density associating to each configuration of your system a weight. Both quantum mechanics/field theory and statistical mechanics/field theory are probabilistic approaches towards doing physics, and the very same tools are used in both. Nevertheless, there are 2 major differences.

1. Statistical mechanics takes a deterministic (classical) theory, and uses probabilities to simplify the system so we can compute things. In quantum mechanics probabilities are inherently fundamental to the system.

2. In statistical mechanics, the Boltzmann factor is well-damped for configurations with large energies, and the partition function is convergent. In quantum mechanics the -1 is replaced by an i, which means instead of damping we have phases. The contribution of each path has the same magnitude, meaning the only way to enhance or suppress a configurations are through superposition of phases: constructive and destructive interference.
   Having no exponential suppression is problematic due to convergence issues. However, we may fix this via analytic continuation, by adding a small negative real part to the exponential:

$$Z = \int D[x(t)] \exp\left((1 + i\epsilon) iS/\hbar\right)$$

   This is akin to the Boltzmann weight in statistical physics. It is by no coincidence that I called this $i\epsilon$: it is the same as the $i\epsilon$ in the Feynman Green's function, and this is how it manifests in the path-integral formalism.

1.2 The Classical Limit

We know that quantum mechanics in the limit of $\hbar \to 0$ gives us classical mechanics. Let us see how this happens in the path-integral formalism, where we expect to recover the principle of least action. This will give us intuition of what it means for a system to be 'quantum', as just how 'quantum' it can be. Our starting point is

$$Z = \int D[x(t)] e^{iS/\hbar}$$

As $\hbar \to 0$, the argument of the exponential blows up. The integrand becomes highly oscillatory with respect to the choice of path (the integration variable), and one expects no contribution from most points of the region of integration, where destructive interference is generic. However, there are stationary points where we can have non-trivial contributions.

To see this very explicitly, we consider a toy model that is much simpler. Here the 'path-integral' is a single integral, and we still have an $\hbar$ factor we can toggle.

$$z = \int_{\mathbb{R}} dx e^{if(x)/\hbar}$$

For instance, we can have $f(x) = (x-2)^2$, and consider the integrand as $\hbar \to 0$. [DRAW THIS] In a general small region $x \in [a, a+\delta]$, f will oscillate rapidly, and the integral over the region will vanish. The exception is if f doesn't change much near a point, i.e. $f'(x_0) = 0$. This is called a stationary point, and in this case all contributions of points near x_0 will interfere constructively. Because f is almost constant at x_0, there is enough space for $e^{if/\hbar}$ to contribute before the destructive interference washes it out. We can make this explicit by Taylor expanding near x_0, using that the linear term vanishes:

$$\begin{aligned}
f(x) &= f(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) + \cdots \\
z &= \int dx \exp\left(\frac{i}{\hbar}(f(x_0) + \frac{1}{2}(x-x_0)^2 + \cdots)\right) \\
&= e^{if(x_0)/\hbar} \int dx \exp\left(\frac{i}{2\hbar}(x-x_0)^2 f''(x_0)\right)\left(1 + \frac{i}{\hbar}\frac{1}{3!}(x-x_0)^3 f'''(x_0) + \cdots\right) \\
&= e^{if(x_0)/\hbar}\sqrt{\frac{2\pi i\hbar}{f''(x_0)}}(1 + \text{higher order})
\end{aligned}$$

This is non-zero, and scales inversely with the second derivative at x_0, which measures how quickly $f'(x)$ stops being zero and destructive interference kicks in. In the last step we have performed a Gaussian integral and neglected higher-order terms. This is precisely a saddle-point approximation you might have used in statistical mechanics. If there is more than 1 extremum of the action (can be a minima, maxima, or saddle), one must sum over all these contributions to obtain the path-integral.

In the path-integral where we integrate over paths (infinitely many degrees of freedom), this works the same way. But instead of extremizing f, we are extremizing the action S. In the limit as $\hbar \to 0$ we only pick up contributions from paths $x_c(t)$ where the functional derivative vanishes,

$$\frac{\delta S}{\delta x(t)}_{x_c} = 0$$

Much more on functional derivatives later. This equation says that taking the classical limit, the only contributing paths are those which don't vary the action at first order. This statement is precisely the principle of least action.

With this in mind, we can now interpret $\hbar$. When $\hbar$ is small but non-zero, the dominant trajectories are still those extremizing S, but now we have non-negligible contributions from generic paths. Our perfectly deterministic principle of least action has now been smeared by quantum fluctuations, giving rise to a probabilistic picture. In the opposite limit where $\hbar \to \infty$ the exponential is just 1 for every path: every path interferes constructively with every other, and quantum fluctuations wash out all classical order. A very general way to think about quantum effects are the fluctuations which smear the determinism of classical mechanics, made necessary by all paths contributing to the path-integral. The strength of these fluctuations is controlled by $\hbar$, a constant intrinsic to the theory. This is exactly like in statistical mechanics, where the analogous object to $\hbar$ is $k_B T$. At $T = 0$ your system is in a state which minimizes the energy with probability 1. Finite temperatures induce thermal fluctuations which smear out this order, and as $T \to \infty$ each microstate has the same contribution to the partition funcition: thermal fluctuations wash out all order, and your system has maximal entropy.

2 Gaussian Integrals

In the above, we have used a saddle point approximation to explicitly compute a 'path-integral' with a single variable of integration. Of critical importance was the Gaussian integrand, which allowed us to perform the integral in the first place. Generically in quantum mechanics and QFT, the path-integrals of all free theories can be reduced to computing an infinite number of Gaussian integrals. Even when we include interactions, we Taylor expand the theory around a free theory, which gives us again Gaussian integrals everywhere. Both here and in SFT the role of the Gaussian integral cannot be overemphasized (see: `https://en.wikipedia.org/wiki/Common_integrals_in_quantum_field_theory`), so in preparation we build up the tools for computing them in finite dimensions. In fact, you can think of finite dimensional Gaussian integrals as free $0+0$-dimensional QFTs. These are often called matrix models.

2.1 Gaussian Integrals in 1 Dimension

The Gaussian distribution in 1 dimension is given by the probability density

$$p(\phi \in \mathbb{R}) = \sqrt{\frac{a}{2\pi}} e^{-a\phi^2/2}$$

The nth moment is

$$\langle \phi^n \rangle = \frac{\int d\phi \phi^n e^{-a\phi^2/2}}{Z(a)}, \qquad Z(a) := \int dx e^{-a\phi^2/2} = \sqrt{\frac{2\pi}{a}}$$

One can think of Z as a primitive partition function or path-integral, but all it does now is act as a normalization. In QFT, these moments will correspond to correlation functions. Odd moments vanish because the integral is odd. We can compute even moments by differentiating the normalization integral with respect to a:

$$\langle \phi^{2n} \rangle = \frac{1}{I(a)} \frac{\partial^n}{\partial(-\frac{a}{2})^n} I(a) = \frac{(-2)^n}{I(a)} \frac{\partial^n}{\partial a^n} \int d\phi e^{-a\phi^2/2} = \frac{(2n-1)!!}{a^n}$$

Alternatively we can make use of the generating functional approach, which generalizes nicely to higher (and infinite) dimensions. We introduce the generating functional

$$Z(J) = \int d\phi e^{-a\phi^2/2 + J\phi} = \sqrt{\frac{2\pi}{a}} e^{J^2/2a}, \qquad \langle x^n \rangle = \frac{1}{Z(0)} \frac{\partial^n}{\partial J^n}_{J=0} Z(J)$$

We can explicitly compute J by completing the square (a very important tool when working with path-integrals). In the other equation, each J-derivative brings a power of x down into the integrand: taking n derivatives and setting $J = 0$, we recover the n-th moment.
Now we compute the $2n$th moment using the expression for J. The first J-derivative of Z brings down a power of J/a. Since we set $J = 0$ at the end of the computation, one of the remaining $2n - 1$ derivatives must hit the factor of J. This gives the recursion relation

$$\langle \phi^{2n} \rangle = \frac{2n-1}{a} \langle \phi^{2(n-1)} \rangle \qquad \Rightarrow \qquad \langle \phi^n \rangle = \frac{(n-1)!!}{a}$$

2.2 Gaussian Integrals in n Dimensions

The Gaussian distribution in n dimensions is given by the probability density

$$p(\phi_i \in \mathbb{R}^n) = \left(\frac{\det A}{2\pi} \right)^{n/2} e^{-\frac{1}{2}\phi_i A_{ij} \phi_j}, \qquad A^\intercal = A$$

For instance, if $A = a\mathbb{1}$ we have $p(\phi) = \left(\frac{a}{2\pi}\right)^{n/2} e^{-\frac{1}{2}\phi_i \phi_i}$. To compute moments we define the generating functional

$$Z(J) = \int d^n\phi e^{-\frac{1}{2}\phi_i A_{ij} \phi_j + J_i \phi_i}, \qquad \langle \phi_{i_1} \cdots \phi_{i_k} \rangle = \frac{1}{Z(0)} \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_k}}_{J=0} Z(J)$$

We can compute the generating functional by completing the square for matrices:

$$-\frac{1}{2}\phi_i A_{ij} \phi_j + J_i \phi_i = -\frac{1}{2} \left(\phi_i A_{ij} \phi_j - 2 J_i \phi_i \right) = (\phi - A^{-1}J)_i A_{ij} (\phi - A^{-1}J)_j - J_i A^{-1}_{ij} J_j$$

$$Z(J) = \int d^n\phi \exp\left(-\frac{1}{2} \left(\phi_i A_{ij} \phi_j - J_i A^{-1}_{ij} J_j \right) \right) = \frac{(2\pi)^n}{\det A} e^{\frac{1}{2} J_i A^{-1}_{ij} J_j}$$

This is a useful trick. Now we can proceed as the 1D case to calculate correlators. The first derivative $\frac{\partial}{\partial J_{i_k}}$ brings down $A_{i_k j} J_j$, using that A is symmetric. One of the remaining derivatives must eliminate the J_j term. Summing over all possibilities one has

$$\langle \phi_{i_1} \cdots \phi_{i_k} \rangle = \sum_{\text{Wick}} \langle \phi_a \phi_b \rangle \cdots \langle \phi_c \phi_d \rangle = \sum_{\text{Wick}} A^{-1}_{ab} \cdots A^{-1}_{cd}$$

The sum is over Wick constractions of the indices $\{i_1, \ldots, i_k\}$. This is the set of all pairs we can form out of the indices. A pair is often denoted by a connector, for instance:

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle = \langle \overbrace{\phi_1 \phi_2} \overbrace{\phi_3 \phi_4} \rangle + \langle \overbrace{\phi_1 \phi_2 \phi_3 \phi_4} \rangle + \langle \overbrace{\phi_1 \phi_2 \phi_3 \phi_4} \rangle = A^{-1}_{12} A^{-1}_{34} + A^{-1}_{13} A^{-1}_{24} + A^{-1}_{14} A^{-1}_{23}$$

For instance, if our Gaussian has $A = a\mathbb{1}$, i.e. $p(x) \propto e^{-\frac{1}{2} x_i x_i}$, then

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle = a(\delta_{12}\delta_{34} + \delta_{13}\delta_{24} + \delta_{14}\delta_{23})$$

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Spring 2023