MIT 18.335, Fall 2006: Review Problems, Solutions

1. (Trefethen/Bau 6.5)

If P is nonzero, then there is an x such that Px is nonzero. For any such x, we have that

$$\|Px\|_2 = \|P^2x\|_2 \leq \|P\|_2\|Px\| \quad \Rightarrow \quad \|P\|_2 \geq 1$$

If P is an orthogonal projector, its singular values are 1 and 0. In particular, $\sigma_1 = \|P\|_2 = 1$. Finally, if $\|P\|_2 = 1$ we show that P must be orthogonal. Consider two vectors $u \in \text{range}(P)$ and $v \in \text{null}(P)$. Decompose u orthogonally into a component along v and a rest r:

$$u = \frac{v^*u}{v^*v}v + r$$

Since $Pr = Pu = u$ and v, r are orthogonal we get

$$\|Pr\|_2^2 = \|u\|_2^2 = \left|\frac{v^*u}{v^*v}\right|^2 \|v\|_2^2 + \|r\|_2^2$$

But when $\|P\|_2 = 1$ it must also be true that

$$\|Pr\|_2^2 \leq \|P\|_2^2\|r\|_2^2 = \|r\|_2^2$$

which is only possible if $v^*u = 0$, that is, u and v are orthogonal and P is an orthogonal projector.

2. (Trefethen/Bau 7.2)

Because of the special structure of A, the odd columns of $\hat{Q}$ will span the same space as the odd columns of A and similarly for the even columns. Therefore, an odd column of A is a linear combination of odd columns of $\hat{Q}$ (only), and similarly for an even column. This means that $\hat{R}$ will be an upper-triangular matrix with $r_{ij} = 0$ when $i + j$ is odd.

3. (Trefethen/Bau 10.1a)

Let F be the Householder reflector for v:

$$F = I - 2\frac{vv^*}{v^*v}$$

The eigenvalue λ_1 corresponding to the eigenvector $x_1 = v$ is then

$$\begin{aligned} Fx_1 &= Fv = v - 2\frac{vv^*}{v^*v}v = -v = \lambda_1 v \\ \lambda_1 &= -1 \end{aligned}$$

And all the other eigenvalues λ_i corresponding to the eigenvectors x_i, which are orthogonal to v, are:

$$\begin{aligned} Fx_i &= x_i - 2\frac{vv^*}{v^*v}x_i = x_i = \lambda_i v \\ \lambda_i &= 1 \end{aligned}$$

The geometric interpretation of this is as follows. The reflector F reflects the space $\mathbb{C}^n$ across the hyperplane H orthogonal to v. The reflection of v is $-v$, so this must be an eigenvector with the eigenvalue -1. All the other eigenvectors x_i are in H, and are not affected by the reflection, that is, $Fx_i = x_i$. This means that all the other eigenvalues are 1.

(Trefethen/Bau 10.1b)

$$\det(F) = \prod_{i=1}^{n} \lambda_i(F) = -1$$

(Trefethen/Bau 10.1c)

The singular values of F are the eigenvalues of F^*F:

$$\begin{aligned} F^*F &= \left(I - 2\frac{vv^*}{v^*v}\right)^* \left(I - 2\frac{vv^*}{v^*v}\right) \\ &= I - 4\frac{vv^*}{v^*v} + 4\frac{vv^*vv^*}{(v^*v)^2} = I \end{aligned}$$

which are all 1.

4. **(Trefethen/Bau 15.1e)**

First, we note from the Taylor expansion of e^x that in exact arithmetic, if n terms are included in the sum, the terms neglected $\sum_{k=n+1}^{\infty} 1/k!$ are bounded by $O(\epsilon_{\text{machine}})$. Also, since there are no inputs x to this problem, we can not have backward stability unless $\tilde{f} = f$, which is impossible since $f = e$. Instead we consider the forward error $|\tilde{f} - f|$ to determine if the algorithm is stable or unstable.

The factorial $1/k!$ can be computed with a relative accuracy of $2k\epsilon_{\text{machine}} + O(\epsilon_{\text{machine}}^2)$, since it has k roundings to floating points, $k-1$ floating point multiplications, and 1 floating point division. Now, the sum $\tilde{f}$ computed from the left can be written (neglecting second order terms)

$$\left(\cdots\left(\left(\frac{1}{0!}(1 + 2\cdot 0\cdot \epsilon_1) + \frac{1}{1!}(1 + 2\cdot 1\cdot \epsilon_2)\right)(1+\epsilon_3) + \frac{1}{2!}(1 + 2\cdot 2\cdot \epsilon_4)\right)(1+\epsilon_5)\cdots\right)$$

where $|\epsilon_i| \le \epsilon_{\text{machine}}$. The largest error is obtained if all $\epsilon_i = \epsilon_{\text{machine}}$, in which case the coefficient a_1 for all first powers of $\epsilon_{\text{machine}}$ is:

$$a_1 = \sum_{k=0}^{n}\left(\frac{2k+1}{k!} + \sum_{j=0}^{k-1}\frac{1}{j!}\right)$$

But this coefficient grows with increasing number of terms n, a simple lower bound is:

$$a_1 \ge \sum_{k=1}^{n} \frac{1}{0!} = n$$

Therefore, since n grows with decreasing $\epsilon_{\text{machine}}$, the error cannot be bounded as $O(\epsilon_{\text{machine}})$ and the algorithm is unstable.

(Trefethen/Bau 15.1f)

In this case, the algorithm computes

$$\left(\cdots\left(\left(\frac{1}{n!}(1 + 2n\epsilon_1) + \frac{1}{(n-1)!}(1 + 2(n-1)\epsilon_2)\right)(1+\epsilon_3) + \frac{1}{(n-2)!}(1 + 2(n-2)\epsilon_4)\right)(1+\epsilon_5)\cdots\right)$$

and the coefficient a_1 for all first powers of ϵ_i becomes

$$a_1 = \sum_{k=0}^{n} \left(\frac{2(n-k)+1}{(n-k)!} + \sum_{j=0}^{k-1} \frac{1}{(n-j)!} \right)$$

This time, the coefficient can be bounded from above independently of n:

$$a_1 = 2\sum_{k=0}^{n} \frac{n-k}{(n-k)!} + \sum_{k=0}^{n} \frac{1}{(n-k)!} + \sum_{k=0}^{n} \frac{n-k}{(n-k)!} = 3\sum_{k=0}^{n-1} \frac{1}{(n-k-1)!} + \sum_{k=0}^{n} \frac{1}{(n-k)!} \leq 4e$$

Therefore, the total relative error is bounded by $O(\epsilon_{\text{machine}})$, and the algorithm is stable.

5. (Trefethen/Bau 21.3)

1. Since A is nonsingular, it must have some nonzero entry in row 1, let's say in column j. Let Q_1 swap columns 1 and j. The upper left element of AQ_1 will then be nonzero, and one step of Gaussian elimination can be performed. The upper left element of U will then also be nonzero, and the lower right $(m-1)\times(m-1)$ submatrix will be nonsingular just like A. This process can be applied recursively to this submatrix, and with the same syntax as in the book, the matrix is factorized as

$$L_{m-1} \cdots L_2 L_1 A Q_1 Q_2 \cdots Q_{m-1} = U$$

   or, equivalently, with $Q = Q_1 Q_2 \cdots Q_{m-1}$ and $L = (L_{m-1} \cdots L_2 L_1)^{-1}$,

$$AQ = LU$$

2. If A is singular, it is not always possible to change columns so that the upper left element of the matrix is nonzero. The algorithm will then break down, and the matrix cannot be factorized in the form $AQ = LU$.

   An example is the matrix

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

   The columns are equal, so $AQ = A$. But this cannot be factorized as LU, since

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} \end{pmatrix}$$

   Solving gives that $u_{11} = 0$, but then it is not possible to have $l_{21}u_{11} = 1$. Therefore, there is no factorization $AQ = LU$.