Advanced Stochastic Processes.

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LECTURE 6

The Reflection Principle. The Distribution of the Maximum. Brownian motion with drift

Lecture outline

- Quick intro to stopping times
- Reflection principle
- Brownian motion with drift

6.1. Technical preliminary: stopping times

Stopping times are loosely speaking "rules" by which we interrupt the process without looking at the process after it was interrupted. For example "sell your stock the first time it hits $20 per share" is a stopping rule. Whereas, "sell your stock one day before it hits $20 per share" is not a stopping rule, since we do not know the day (if any) when it hits this price.

Given a stochastic process $\{X_t\}_{t\geq 0}$ with $t \in \mathbb{Z}_+$ or $t \in \Re_+$, a a random variable T is called a stopping time if for every time the event $T \leq t$ is completely determined by the history $\{X_s\}_{0\leq s\leq t}$.

This is not a formal definition. The formal definition will be given later when we study *filtration*. Then we will give the definition in terms of the underlying $(\Omega, \mathcal{F}, \mathbb{P})$. For now, though, let us just adopt this loose definition.

6.2. The Reflection principle. The distribution of the maximum

The goal of this section is to obtain the distribution of

$$M(t) \triangleq \sup_{0\leq s\leq t} B(s)$$

for any given t. Surprisingly the resulting expression is very simple and follows from one of the key properties of the Brownian motion – reflection principle.

Given $a > 0$, define

$$T_a = \inf\{t : B(t) = a\}$$

– the first time when Brownian motion hits level a. When no such time exists we define $T_a = \infty$, although we now show that it is finite almost surely.

Proposition 1. $T_a < \infty$ almost surely.

Proof. Note that if B hits some level $b \geq a$ almost surely, then by continuity and since $B(0) = 0$, it hits level a almost surely. Therefore, it suffices to prove that $\limsup_t B(t) = \infty$ almost surely. This in its own order will follow from $\limsup_n B(n) = \infty$ almost surely. □

Problem 1. Prove that $\limsup_n |B(n)| = \infty$ almost surely.
HINT: Use the independent increments property of the Brownian motion and continuity property: $\mathbb{P}(\cap A_j) = \lim_j \mathbb{P}(A_j)$ for monotone decreasing sequences of events A_j, which we established earlier.

The differential property of the Brownian motion suggests that

$$B(T_a + s) - B(T_a) = B(T_a + s) - a \tag{6.1}$$

is also a Brownian motion, independent from $B(t), t \leq T_a$. The only issue here is that T_a is a random instance and the differential property was established for fixed times t. Turns out (we do not prove this) the differential property also holds for a random time T_a since it is a stopping time and is finite almost surely. The first is an immediate consequence of its definition: we can determine whether $T_a \leq t$ by checking looking at the path $B(u), 0 \leq u \leq t$. The almost sure finiteness follows from Proposition 1. The property (6.1) is called the *strong independent increments* property of the Brownian motion.

Theorem 6.2 (The reflection principle). *Given a standard Brownian motion $B(t)$, for every $a \geq 0$*

$$\mathbb{P}(M(t) \geq a) = 2\mathbb{P}(B(t) \geq a) = 2\frac{1}{\sqrt{2\pi t}}\int_a^\infty e^{-\frac{x^2}{2t}}dx. \tag{6.3}$$

Proof. We have

$$\mathbb{P}(B(t) \geq a) = \mathbb{P}(B(t) \geq a, M(t) \geq a) + \mathbb{P}(B(t) \geq a, M(t) < a).$$

Note, however, that $\mathbb{P}(B(t) \geq a, M(t) < a) = 0$ since $M(t) \geq B(t)$. Now

$$\begin{aligned}\mathbb{P}(B(t) \geq a, M(t) \geq a) &= \mathbb{P}(B(t) \geq a | M(t) \geq a)\mathbb{P}(M(t) \geq a) \\ &= \mathbb{P}(B(t) \geq a | T_a \leq t)\mathbb{P}(M(t) \geq a).\end{aligned}$$

We have that $B(T_a + s) - a$ is a Brownian motion. Conditioned on $T_a \leq t$,

$$\mathbb{P}(B(t) \geq a | T_a \leq t) = \mathbb{P}(B(T_a + (t - T_a)) - a \geq 0 | T_a \leq t) = \frac{1}{2}$$

since the Brownian motion satisfies $\mathbb{P}(B(t) \geq 0) = 1/2$ for every t. Applying this identity, we obtain

$$\mathbb{P}(B(t) \geq a) = \frac{1}{2}\mathbb{P}(M(t) \geq a).$$

This establishes the required identity (6.3). □

We now establish the joint probability distribution of $M(t)$ and $B(t)$.

Proposition 2. For every $a > 0, y \geq 0$

$$\mathbb{P}(M(t) \geq a, B(t) \leq a - y) = \mathbb{P}(B(t) > a + y). \tag{6.4}$$

Proof. We have

$$\begin{aligned}\mathbb{P}(B(t) > a + y) &= \mathbb{P}(B(t) > a + y, M(t) \geq a) + \mathbb{P}(B(t) > a + y, M(t) < a) \\ &= \mathbb{P}(B(t) > a + y, M(t) \geq a) \\ &= \mathbb{P}(B(T_a + (t - T_a)) - a > y | M(t) \geq a)\mathbb{P}(M(t) \geq a).\end{aligned}$$

But since $B(T_a + (t - T_a)) - a$, by differential property is also a Brownian motion, then, by symmetry

$$\begin{aligned}\mathbb{P}(B(T_a + (t - T_a)) - a > y | M(t) \geq a) &= \mathbb{P}(B(T_a + (t - T_a)) - a < -y | M(t) \geq a) \\ &= \mathbb{P}(B(t) < a - y | M(t) \geq a).\end{aligned}$$

We conclude

$$\mathbb{P}(B(t) > a + y) = \mathbb{P}(B(t) < a - y | M(t) \geq a)\mathbb{P}(M(t) \geq a) = \mathbb{P}(B(t) < a - y, M(t) \geq a).$$

□

We now compute the Laplace transform of the hitting time T_a.

Proposition 3. For every $\lambda > 0$

$$\mathbb{E}[e^{-\lambda T_a}] = e^{-\sqrt{2\lambda}a}.$$

Proof. We first compute the density of T_a. We have

$$\mathbb{P}(T_a \leq t) = \mathbb{P}(M(t) \geq a) = 2\mathbb{P}(B(t) \geq a) = \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-\frac{x^2}{2t}} dx = 2(1 - N(\frac{a}{\sqrt{t}})).$$

By differentiating with respect to t we obtain that the density of T_a is given as

$$\frac{a}{t^{\frac{3}{2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2t}}.$$

Therefore

$$\mathbb{E}[e^{-\lambda T_a}] = \int_0^\infty e^{-\lambda t} \frac{a}{t^{\frac{3}{2}}\sqrt{2\pi}} e^{-\frac{a^2}{2t}} dt.$$

(Why do we not integrate over the negative part?) Computing this integral is a boring exercise in calculus. We just state the result which is $e^{-\sqrt{2\lambda}a}$. □

6.3. Brownian motion with drift

So far we considered a Brownian motion which is characterized by zero mean and some variance parameter σ^2. The standard Brownian motion is the special case $\sigma = 1$.

There is a natural way to extend this process to a non-zero mean process by considering $B_\mu(t) = \mu t + B(t)$, given a Brownian motion $B(t)$. Some properties of $B_\mu(t)$ follow immediately. For example given $s < t$, the increments $B_\mu(t) - B_\mu(s)$ have mean $\mu(t - s)$ and variance $\sigma^2(t - s)$.

Also, by the Time Reversal property of B (see Lecture 5) we know that $\lim_{t\to\infty}(1/t)B(t) = 0$ almost surely. Therefore, almost surely

$$\lim_{t\to\infty} \frac{B_\mu(t)}{t} = \mu.$$

When $\mu < 0$ this means that $M_\mu(\infty) \triangleq \sup_{t\geq 0} B_\mu(t) < \infty$ almost surely. On the other hand $M_\mu(\infty) \geq 0$ (why?).

Our goal now is to compute the probability distribution of $M_\mu(\infty)$. For simplicity assume that B is the standard Brownian motion. The general case is obtained by dividing $B_\mu(t) = \mu t + B(t)$ by σ and observing that now the driftless part $B(t)/\sigma$ corresponds to the standard Brownian motion.

Theorem 6.5. *For every $\mu < 0$, the distribution of $M_\mu(\infty)$ is exponential with parameter $2|\mu|$. Namely, for every $x \geq 0$*

$$\mathbb{P}(M_\mu(\infty) > x) = e^{-2|\mu|x}.$$

The proof is in Section 6.8 of Resnick's book in the course packet. The proof consists of two parts. We first show that the distribution of $M_\mu(\infty)$ is exponential. Then we compute its parameter.

6.4. Additional reading materials

- Sections 6.5 and 6.8 from Chapter 6 of Resnick's book "Adventures in Stochastic Processes" in the course packet.
- Sections 7.3 and 7.4 in Durrett [**2**].
- Billingsley [**1**], Section 9.

BIBLIOGRAPHY

1. P. Billingsley, *Convergence of probability measures*, Wiley-Interscience publication, 1999.
2. R. Durrett, *Probability: theory and examples*, Duxbury Press, second edition, 1996.