Solution for 18.112 Mid 1

Problem 1.

Solution:

$$z^3 = 8e^{-i\pi/2} \Longrightarrow z = 2e^{-i(\frac{\pi}{6}+\frac{2n\pi}{3})},\ 0 \le n \le 2,$$
$$\Longrightarrow z = 2i \text{ or } z = \sqrt{3} - i \text{ or } z = -\sqrt{3} - i.$$

Problem 2.

Method 1.

$$\begin{aligned}\int_{|z-1|=\frac{1}{2}} \frac{dz}{(1-z)^3} &= \int_0^{2\pi} \frac{ie^i t/2}{(-e^{it}/2)^3} dt \\ &= \frac{-i}{2} \cdot 8 \int_0^{2\pi} e^{-i\cdot 2t} dt \\ &= -4i \left. \frac{e^{-2it}}{-2i} \right|_0^{2\pi} \\ &= 0.\end{aligned}$$

Method 2. Let $f(z) \equiv 1$. By (24) on Page 120, we get

$$0 = f''(1) = \frac{2!}{2\pi i} \int_{|z-1|=\frac{1}{2}} \frac{dz}{(z-1)^3}.$$

Problem 3.

Solution: 1) $\int_{|z|=1} \frac{e^z+z}{z-2} dz = 0$, since $2 \notin \{z : |z| < 1\}$.

2) $\int_{|z|=3} \frac{e^z+z}{z-2} dz = 2\pi i(e^2 + 2)$, since $n(\gamma, 2) = 1$. (Theorem 6 on P119.)

Problem 4.

Solution: Let

$$g(z) = f(\frac{1}{z}), \ \forall z \neq 0,$$

then g is analytic on $\mathbb{C} \setminus \{0\}$, and the singularity at 0 is removable or is pole of order h.

If the singularity of g at 0 is removable, then $\lim_{z \to 0} g(z)$ exists and is finite, i.e. $\lim_{z \to \infty} f(z)$ exists and is finite. Thus f is bounded on $\mathbb{C}$. Since f is analytic and bounded in the whole plane, it is a constant.

If 0 is pole of order h, then

$$g(z) = B_h z^{-h} + B_{h-1} z^{-h+1} + \cdots + B_1 z^{-1} + \phi(z),$$

where $\phi(z)$ is analytic on $\mathbb{C}$. Since f is continuous (analytic) at 0, $\lim_{z \to \infty} g(z)$ exists and is finite. Thus $\lim_{z \to \infty} \phi(z)$ exists and is finite. So $\phi(z)$ is bounded in the whole plane, and thus $\phi(z) = B_0$ is constant. So

$$f(z) = g(\frac{1}{z}) = B_h z^h + B_{h-1} z^{h-1} + \cdots + B_1 z + B_0$$

is polynomial.

Problem 5.

Method 1. Take

$$C : |z| = R, \text{ where } R > 100.$$

For any $m > n$, we have

$$\begin{aligned} |f^{(m)}(0)| &= \left| \frac{m!}{2\pi i} \int_C \frac{f(\xi)\, d\xi}{\xi^{m+1}} \right| \\ &\leq \frac{m!}{2\pi} \left| \int_C \xi^{n-m-1}\, d\xi \right| \\ &= \frac{m!}{2\pi} \frac{R^{n-m}}{n-m} \longrightarrow 0 \text{ as } R \to \infty. \end{aligned}$$

Thus $f^{(m)}(0) = 0$ for any $m > n$. By the Taylor expansion,

$$\begin{aligned} f(z) &= f(0) + \frac{f'(0)}{1!} z + \cdots + \frac{f^n(0)}{n!} z^n + \frac{f^{n+1}(0)}{(n+1)!} z^{n+1} + \cdots \\ &= f(0) + \frac{f'(0)}{1!} z + \cdots + \frac{f^n(0)}{n!} z^n \end{aligned}$$

is polynomial.

Method 2. By $|f(z)| < |z|^n$, we know

$$\lim_{z\to 0} z^{n+1} f(1/z) = 0,$$

i.e. f has a nonessential singularity at ∞. By last problem, f is polynomial.