

Advanced Stochastic Processes.

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LECTURE 19

σ -fields on measure spaces and weak convergence

Lecture outline

- σ -fields on metric spaces.
- Kolmogorov σ -field on $C[0, T]$.
- Weak convergence.

19.1. Borel σ -fields on metric space

We consider a metric space (S, ρ) . To discuss probability measures on metric spaces we first need to introduce σ -fields.

Definition 19.1. Borel σ -field \mathcal{B} on S is the field generated by the set of all open sets $U \subset S$.

Lemma 19.2. *Suppose S is Polish. Then every open set $U \subset S$ can be represented as a countable union of balls $B(x, r)$.*

Proof. Since S is Polish, we can identify a countable set x_1, \dots, x_n, \dots which is dense. For each $x_i \in U$, since U is open we can find a radius r_i such that $B(x_i, r_i) \subset U$. Consider $r_i^* = \arg \sup\{r : B(x_i, r) \subset U\} > 0$ and set $r_i = r_i^*/2$. Then $\cup_i B(x_i, r_i) \subset U$. In order to finish the proof we need to show that equality $\cup B(x_i, r_i) = U$ holds. Consider any $x \in S$. There exists r such that $B(x, r) \subset U$. Since set (x_i) is dense there exists x_k such that $\rho(x_k, x) < r/4$. Then, by triangle inequality, $B(x_k, 3r/4) \subset B(x, r) \subset U$. This means, by definition of r_k^* and r_k , that $r_k \geq 3r/8$. In particular, $B(x_k, 3r/8) \subset U$. But $x \in B(x_k, 3r/8)$ since the distance $\rho(x_k, x) \leq r/4$. This shows $x \in B(x_k, r_k)$. Since x was arbitrary, then $U \subset \cup_i B(x_i, r_i)$. \square

From this lemma we obtain immediately that

Corollary 1. Suppose S is Polish. Then \mathcal{B} is generated by the set of all balls $B(x, r), x \in S, r \geq 0$.

Note that this definition is the definition of Borel σ -field on \mathbb{R} and \mathbb{R}^d as well as Borel σ -field on $C[0, T]$, as all of these spaces are Polish. Thus we have found a unifying definition of Borel

σ -field. This applies to the space $C[0, \infty)$ with metric $\|\cdot\| = \sum_k \frac{1}{2^k} \|\cdot\|_k$, but this needs some work to show.

An immediate implication of Definition 19.1 and Problem 5 from previous lecture is as follows.

Proposition 1. Every continuous function $f : S_1 \rightarrow S_2$ is measurable.

We may now revisit some of the measurability results we established by brute force in our first lecture. Recall that we established in Proposition 3, Lecture 1 that a) the set of non-negative functions $x(t) \geq 0$, b) the set of functions x with $\lim_{t \rightarrow \infty} x(t) = \infty$, c) the set of functions $x(t) \leq t$ are all Borel measurable. We now see that these follow almost immediately from the definition. Consider $x \geq 0$ first restricted on $[0, T]$. This set is closed. Indeed if $x(t) \leq -\epsilon < 0$ then $B(x, \epsilon/2)$ does not contain any elements y such that $y \geq 0$. Since it is closed it is measurable. For $C[0, \infty)$ we obtain the set of $x \geq 0$ as intersection of all x which are non-negative on $[0, n]$, $n = 1, 2, \dots$. For the set of all $x(t) \leq t$, we obtain similarly that this set is closed on each $[0, n]$ and therefore measurable. The proof for b) remains involved as this set is neither closed nor open (show this).

Recall also our proof that the function $f : [0, 1]^2 \rightarrow \mathbb{R}$ defined by $f(x, y) = x + y$ is measurable. From Proposition 1 this follows immediately, since this function is continuous.

Problem 1. Given a metric space (S, ρ) and $x \in S$, consider function $f(y) = \rho(x, y)$. Show that $f : S \rightarrow \mathbb{R}$ is uniformly continuous and therefore measurable. Use this to deduce that in $C[0, \infty)$ every ball $B(x, r)$ corresponding to the metric $\|\cdot\| = \sum_k \frac{1}{2^k} \|\cdot\|_k$ is measurable w.r.t. Borel σ -field \mathcal{B} . (Recall that we defined \mathcal{B} as the field generated by balls $B(x, r, T) = \{y : \|x - y\|_T \leq r\}$).

19.2. Kolmogorov σ -field on $C[0, T]$

Now let us focus on $C[0, T]$ and the Borel σ field \mathcal{B} on it. For each $t \geq 0$ define a projection $\pi_t : C[0, T] \rightarrow \mathbb{R}$ as $\pi_t(x) = x(t)$. Observe that π_t is a uniformly continuous mapping. Indeed

$$|\pi_t(x) - \pi_t(y)| = |x(t) - y(t)| \leq \|x - y\|.$$

This immediately implies continuity.

The family of projection mappings π_t give rise to an alternative field.

Definition 19.3. The Kolmogorov σ -field \mathcal{K} on $C[0, T]$ is the σ -field generated by $\pi_t^{-1}(B)$, $t \in [0, T]$, $B \in \mathcal{B}$, where \mathcal{B} is the Borel field of \mathbb{R} .

It turns out (and this will be useful) that the two fields are identical:

Theorem 19.4. The Kolmogorov σ field \mathcal{K} is identical to the Borel field \mathcal{B} of $C[0, T]$.

Proof. First we show that $\mathcal{K} \subset \mathcal{B}$. Since π_t is continuous then, applying Problem 5 from previous lecture, for every open set $U \subset \mathbb{R}$ $\pi_t^{-1}(U)$ is open in $C[0, T]$. This applies to all open intervals U . Thus each $\pi_t^{-1}(U) \in \mathcal{B}$. This shows $\mathcal{K} \subset \mathcal{B}$. Since $C[0, T]$ is Polish, then by Corollary 1, it suffices to check that every ball $B(x, r) \in \mathcal{K}$. Fix $x \in C[0, T]$, $r \geq 0$. For each rational $q \in [0, T]$, consider $B_q \triangleq \pi_q^{-1}([x(q) - r, x(q) + r])$. This is the set of all functions y such that $|y(q) - x(q)| \leq r$. Consider $\bigcap_q B_q$. As a countable intersection, this is a set in \mathcal{K} . We claim that $\bigcap_q B_q = B$. This implies the result.

To establish the claim, note that $B \subset B_q$ for each q . Now suppose $y \notin B$. We will show that $y \notin \bigcap_q B_q$ and we are done. Namely, for some $t \in [0, T]$ we have $|y(t) - x(t)| \geq r + \delta > r$. Find a sequence of rational values q_n converging to t . By continuity of x, y we have $x(q_n) \rightarrow x(t)$, $y(q_n) \rightarrow y(t)$. Therefore for all sufficiently large n we have $|y(q_n) - x(q_n)| > r$. This means $y \notin \bigcap_q B_q$. \square

19.3. Weak convergence

We now turn to a very important concept of weak convergence or convergence of probability measures. In Lecture 8 we defined a convergence of a sequence of random variables $X_n \Rightarrow X$ via their distribution functions. Now consider random variables $X : \Omega \rightarrow S$ which take values in some metric space (S, ρ) . Again we define X to be a random variable if X is a measurable transformation. We would like to give a meaning to $X_n \Rightarrow X$. In order to do this we first define convergence of probability measures.

Given a metric space (S, ρ) and the corresponding Borel σ -field \mathcal{B} , suppose we have a sequence $\mathbb{P}, \mathbb{P}_n, n = 1, 2, \dots$ of probability measures on (S, \mathcal{B}) . Let $C_b(S)$ denote the set of all continuous bounded real valued functions on S . In particular every function $X \in C_b(S)$ is measurable.

Theorem 19.5 (Portmentau Theorem). *The following conditions are equivalent.*

- (a) $\lim_n \mathbb{E}_{\mathbb{P}_n}[X] = \mathbb{E}_{\mathbb{P}}[X], \forall X \in C_b(S)$.
- (b) *For every closed set $F \subset S$, $\limsup_n \mathbb{P}_n(F) \leq \mathbb{P}(F)$.*
- (c) *For every open set $U \subset S$, $\liminf_n \mathbb{P}_n(U) \geq \mathbb{P}(U)$.*
- (d) *For every set $A \in \mathcal{B}$ such that $\mathbb{P}(\partial A) = 0$, the convergence $\lim_n \mathbb{P}_n(A) = \mathbb{P}(A)$ holds.*

What is the meaning of this theorem? Ideally we would like to say that a sequence of measures \mathbb{P}_n converges to measure \mathbb{P} if for every set A , $\mathbb{P}_n(A) \rightarrow \mathbb{P}(A)$. However, this turns out to be too restrictive. Recall our example from Lecture 8 where $X_n \Rightarrow X$ for some X_n, X but their distribution functions did not converge at some discontinuity point. For a similar reason we have restrictions here. In some sense the fourth part of the theorem is the meaningful part. Second and third are technical. The first is a very useful implication.

Definition 19.6. A sequence of measures \mathbb{P}_n is said to converge weakly to \mathbb{P} if one of the four equivalent conditions in Theorem 19.5 holds.

Proof. (a) \Rightarrow (b) Consider a closed set F . For any $\epsilon > 0$, define $F^\epsilon = \{x \in S : \rho(x, S) \leq \epsilon\}$, where the distance between the sets is defined as the smallest distance between any two points in them. Let us first show that $\cap_\epsilon F^\epsilon = F$. The inclusion $\cap_\epsilon F^\epsilon \supset F$ is immediate. For the other side, consider any $x \notin F$. Since $S \setminus F$ is open, then $B(x, r) \subset S \setminus F$ for some r . This means $x \notin F^r$ and the assertion is established.

Invoking the continuity theorem, the equality $\cap_\epsilon F^\epsilon = F$ implies

$$(19.7) \quad \lim_{\epsilon \rightarrow 0} \mathbb{P}(F^\epsilon) = \mathbb{P}(F).$$

Define $X_\epsilon : S \rightarrow [0, 1]$ by $X_\epsilon(x) = (1 - \rho(x, F)\epsilon^{-1})^+$. Then $X_\epsilon = 1$ on F and $X_\epsilon = 0$ outside of F^ϵ . Specifically,

$$1\{F\} \leq X_\epsilon \leq 1\{F^\epsilon\},$$

which implies

$$\mathbb{P}_n(F) \leq \mathbb{E}_{\mathbb{P}_n}[X_\epsilon] \leq \mathbb{P}_n(F^\epsilon), \quad \mathbb{P}(F) \leq \mathbb{E}_{\mathbb{P}}[X_\epsilon] \leq \mathbb{P}(F^\epsilon),$$

Note that X_ϵ is a continuous function (this requires check). Therefore, by assumption $\lim_n \mathbb{E}_{\mathbb{P}_n}[X_\epsilon] = \mathbb{E}[X_\epsilon]$. Combining, we obtain that

$$\limsup_n \mathbb{P}_n(F) \leq \limsup_n \mathbb{E}_{\mathbb{P}_n}[X_\epsilon] = \lim_n \mathbb{E}_{\mathbb{P}_n}[X_\epsilon] = \mathbb{E}_{\mathbb{P}}[X_\epsilon] \leq \mathbb{P}(F^\epsilon)$$

Finally, using (19.7) we conclude that $\limsup_n \mathbb{P}_n(F) \leq \mathbb{P}(F)$.

(b) \Rightarrow (c) This part follows immediately by observing that $F = U^c$ is a closed set and $\mathbb{P}(F) = 1 - \mathbb{P}(U)$, $\mathbb{P}_n(F) = 1 - \mathbb{P}_n(U)$. Moreover (c) \Rightarrow (d).

(c) \Rightarrow (d) Given any set A , consider its closure $\bar{A} \supset A$ and interior $A^\circ \subset A$. Applying (b) and (c), we have

$$\mathbb{P}(\bar{A}) \geq \limsup_n \mathbb{P}_n(\bar{A}) \geq \limsup_n \mathbb{P}_n(A) \geq \liminf_n \mathbb{P}_n(A) \geq \liminf_n \mathbb{P}_n(A^\circ) \geq \mathbb{P}(A^\circ).$$

Now if $\mathbb{P}(\partial A) = \mathbb{P}(\bar{A} \setminus A^\circ) = 0$, then we obtain an equality across and, in particular, $\mathbb{P}_n(A) \rightarrow \mathbb{P}(A)$.

(d) \Rightarrow (a) Sketch of proof. Let $X \in C_b(S)$. We may assume w.l.g that $0 \leq X \leq 1$. Then $\mathbb{E}_{\mathbb{P}_n}[X] = \int_0^1 \mathbb{P}_n(X > t) dt$, $\mathbb{E}_{\mathbb{P}}[X] = \int_0^1 \mathbb{P}(X > t) dt$. Since X is continuous, for any set $A(t) = \{x : X(x) > t\}$, its boundary satisfies $\partial A \subset 1\{X = t\}$ (exercise). Since probability of the set $1\{X = t\}$ is zero except for countably many points t , then for almost all t , the set $A(t)$ is a continuity set, i.e., $\mathbb{P}(\partial A) = 0$ and hence $\mathbb{P}_n(A(t)) \rightarrow \mathbb{P}(A(t))$. Since also $0 \leq \mathbb{P}, \mathbb{P}_n \leq 1$, then using the Bounded Convergence Theorem applied to "random" t , we obtain that

$$\int_0^1 \mathbb{P}_n(A(t)) dt \rightarrow \int_0^1 \mathbb{P}(A(t)) dt.$$

This implies the result. □

Using the definition of weak convergence of measures we can define weak convergence of metric space valued random variables.

Definition 19.8. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a metric space (S, ρ) , a sequence of measurable transformations $X_n : \Omega \rightarrow S$ is said to converge weakly or in distribution to a transformation $X : \Omega \rightarrow S$, if the probability measures on (S, ρ) generated by X_n converge weakly to the probability measure generated by X on (S, ρ) .

19.4. Additional reading materials

- Notes distributed in the class, Chapter 5.
- Billigsley [1] Chapter 1, Section 2.

BIBLIOGRAPHY

1. P. Billingsley, *Convergence of probability measures*, Wiley-Interscience publication, 1999.