

# Advanced Stochastic Processes.

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## LECTURE 4 Properties of the distribution function $G$

We started with a distribution function  $F(x)$  and considered a modified distribution function

$$G(x) = M(\theta^*)^{-1} \int_{-\infty}^x e^{\theta^* y} dF(y).$$

We showed that  $G$  is indeed a probability distribution function. Let  $\tilde{X}$  be a random variable distributed according to  $G$ .

**Lemma 4.1.** *The following holds*

- (a)  $\mathbb{E}[\tilde{X}] = a$ .
- (b)  $\mathbb{E}[\tilde{X}^2] < \infty$ .

**Proof.** By definition

$$\begin{aligned} \theta^* &= \arg \sup_{\theta} (a\theta - \log M(\theta)) \\ &= \arg \inf_{\theta} (-a\theta + \log M(\theta)) \\ &= \arg \inf_{\theta} (e^{-a\theta + \log M(\theta)}) \\ &= \arg \inf_{\theta} (M(\theta)e^{-a\theta}) \\ &= \arg \inf_{\theta} \mathbb{E}[e^{\theta(X_1 - a)}] \end{aligned}$$

Since  $0 < \theta^* < \theta_0$ , then the derivative of  $\mathbb{E}[e^{\theta(X_1 - a)}]$  at  $\theta^*$  must be equal to zero. Taking derivative at  $\theta^*$ , we conclude

$$\mathbb{E}[(X_1 - a)e^{\theta^*(X_1 - a)}] = 0$$

Multiplying both parts by  $e^{\theta^* a}$  we obtain

$$\mathbb{E}[(X_1 - a)e^{\theta^* X_1}] = 0,$$

implying

$$\mathbb{E}[X_1 e^{\theta^* X_1}] = a \mathbb{E}[e^{\theta^* X_1}] = aM(\theta^*).$$

But

$$\begin{aligned}\mathbb{E}[X_1 e^{\theta^* X_1}] &= \int_{-\infty}^{\infty} y e^{\theta^* y} dF(y) \\ &= M(\theta^*) M^{-1}(\theta^*) \int_{-\infty}^{\infty} y e^{\theta^* y} dF(y) \\ &= M(\theta^*) \int_{-\infty}^{\infty} y dG(y) \\ &= M(\theta^*) \mathbb{E}[\tilde{X}_1]\end{aligned}$$

Therefore  $\mathbb{E}[\tilde{X}_1] = a$ .

The second part  $\mathbb{E}[\tilde{X}_1^2] < \infty$  will be discussed during the recitation.

□