

Advanced Stochastic Processes.

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LECTURE 16

Applications of Ito calculus to finance

Lecture outline

- Trading strategies
- Black-Scholes option pricing formula

16.1. Security price processes, trading strategies and arbitrage

As early as 1900, Louis Bachelier had proposed using Brownian motion as a model for security prices. As insightful as it was, this model is somewhat restrictive. For example it allows stock price to be negative.

A more appropriate and accepted model is to assume that stock prices follow a general Ito process $X(t) = X(0) + \int_0^t d\mu(t)dt + \int_0^t \sigma(t)dB(t)$, where μ, σ are adapted processes. Suppose we make several investment decisions about this stock. In particular at times $t_1 < t_2 < \dots < t_k$ we decide to hold $\theta_k = \theta(t_k)$ stocks of this security. What would be our gain/loss at some later time $t > t_k$? It is simply $\sum_{j \leq k} \theta_j (X(t_{j+1}) - X(t_j))$. It makes sense to assume that our trading strategy θ is an adapted process (including the possibility that θ is deterministic), since otherwise it means we can predict the market. Thus we can think of θ as a simple adapted process. The simplicity means that we make only k trading decisions. But there is no reason to bound the number of trading decisions a priori (unless we take trading fees into account). So we may think of θ as an arbitrary adapted process $\theta_t \in \mathcal{F}_t$. For technical reasons we do assume that $\theta \in H^2$. This is needed so that we exclude the "doubling" strategy pathology which we saw earlier brings a positive gain with probability one.

Definition 16.1. A trading strategy θ_t is an adapted process in H^2 . The gain produced by the trading strategy θ_t during the time interval $[0, T]$ is defined to be

$$(16.2) \quad \int_0^T \theta(t) dX(t) = \int_0^T \theta(t) \mu(t) dt + \int_0^T \theta(t) \sigma(t) dB(t)$$

Given a vector of securities $\bar{X}(t) = (X_1(t), \dots, X_m(t))$ a vector of trading strategies $\bar{\theta}(t) = (\theta_1(t), \dots, \theta_m(t))$ into securities $\bar{X}(t)$ is defined to be *self-financing* if for every time instance t

$$(16.3) \quad \theta(t) \cdot X(t) = \theta(0) \cdot X(0) + \int_0^t \theta(s) \cdot dX(s)$$

$$(16.4) \quad = \sum_{1 \leq j \leq m} \theta_j(0) X_j(0) + \sum_{1 \leq j \leq m} \int_0^t \theta_j(s) dX_j(s).$$

This second part of the definition simply means that whatever we have at any time we invest. This is easier to understand when we have a simple strategy, that is $\theta \in S^2$. Then we trade at times $0 = t_0 < t_1 < \dots < t_n = t$. We start with portfolio $\bar{\theta}(0)$ buy $\bar{\theta}(0) \cdot \bar{X}(0)$ worth of dollars of a security. At time t_1 our portfolio is worth $\bar{\theta}(0) \cdot \bar{X}(0) + \bar{\theta}(0) \cdot (\bar{X}(t_1) - \bar{X}(0)) \triangleq W(t_1)$. We create some other portfolio $\bar{\theta}(t_1)$ with the condition that $\bar{\theta}(t_1) \cdot \bar{X}(t_1) = W(t_1)$ (self-financing). At time t_2 our portfolio is worth

$$\begin{aligned} W(t_1) + \bar{\theta}(t_1) \cdot (\bar{X}(t_2) - \bar{X}(t_1)) &= \bar{\theta}(0) \cdot \bar{X}(0) + \bar{\theta}(0) \cdot (\bar{X}(t_1) - \bar{X}(0)) + \bar{\theta}(t_1) \cdot (\bar{X}(t_2) - \bar{X}(t_1)) \\ &\triangleq W(t_2). \end{aligned}$$

Then we create portfolio $\bar{\theta}(t_2)$ with the condition $\bar{\theta}(t_2) \cdot \bar{X}(t_2) = W(t_2)$, and so on. In the end we obtain

$$\begin{aligned} \bar{\theta}(t) \cdot \bar{X}(t) = W(t) &= \bar{\theta}(0) \cdot \bar{X}(0) + \sum_{j \leq n-1} \bar{\theta}(t_j) \cdot (\bar{X}(t_{j+1}) - \bar{X}(t_j)) \\ &= \bar{\theta}(0) \cdot \bar{X}(0) + \int_0^t \bar{\theta}(s) d\bar{X}(s). \end{aligned}$$

We do not require that $\theta \geq 0$. Having a negative $\theta < 0$ means essentially short-selling the security.

One of the basic assumptions of classical financial models is that the security market does not allow *arbitrage*. Arbitrage is an opportunity of obtaining a positive gain without any risk over some time period $[0, T]$. Mathematically, we define it as follows.

Definition 16.5. A vector of trading strategies $\bar{\theta}(t)$ in securities $\bar{X}(t)$ is defined to be arbitrage if $\bar{\theta}(0) \cdot \bar{X}(0) < 0$ and $\bar{\theta}(T) \cdot \bar{X}(T) \geq 0$ a.s., or $\bar{\theta}(0) \cdot \bar{X}(0) \leq 0$ and $\bar{\theta}(T) \cdot \bar{X}(T) > 0$ (where we say $Z > 0$ means $Z \geq 0$ and $\mathbb{P}(Z > 0) > 0$).

We see that in either case an arbitrage creates an opportunity to gain money without any losses. In the first case first case we gain $-\theta(0)X(0)$ with probability one. In the second case we gain $\theta(T)X(T)$ with a positive probability. It is a basic assumption in finance theory that market prices are such that there does not exist a trading strategy creating an arbitrage opportunity. As it turns out, under technical assumptions, this is equivalent to existence of a change of measure such that with respect to the new measure X is a martingale.

16.2. Black-Scholes option pricing formula

For the purposes of this section we assume that we are dealing with two securities:

- (a) A stock, whose time t price $S(t)$ follows a *geometric Brownian motion*

$$S(t) = \exp(\alpha t + \sigma B(t))$$

for some constants α, σ ; and

(b) A bond, whose time t price $\beta(t)$ at time t is given as

$$\beta(t) = \beta(0) \exp(rt)$$

for some constants $\beta(0), r$.

Both of these are Ito processes. The stock process we find using Ito formula is

$$(16.6) \quad dS = S(\alpha dt + \sigma dB) + \frac{1}{2} S \sigma^2 (dB)^2 = S(\mu dt + \sigma dB),$$

where $\mu = \alpha + \frac{\sigma^2}{2}$. The β process is simply a deterministic process satisfying $d\beta = \beta r dt$. μ is called the *instantaneous rate of return*, σ is called the *volatility* and r is called the *risk free interest rate*.

In addition suppose we have the following *derivative security* called option. Specifically, we will consider a so called European call option which is parametrized by a certain *strike price* K and maturity time T . A European call option gives its owner the right to buy the stock S at time T for a price K . Thus the payoff to the owner of the option is $\max(S(T) - K, 0)$ at time T (the option will not be exercised when the price $S(T) < K$ to avoid a loss). The main question is what should be the price of this stock at time 0? One would expect the fair price of the option to depend on the owner's/seller's perception of the risk of the stock. The main insight of the Black Scholes was to realize that the price of the option is tied to the price of the stock in a *deterministic* way if there is to be no arbitrage.

The reason for this deterministic dependence has to do with the fact that one can simply "replicate" the option by carefully constructing a portfolio consisting of a bond and a stock. Another surprising aspect of the Black Scholes result was that the portfolio and the price can be computed in a very explicit form.

We first present the Black Scholes formula and then discuss its relevance to option pricing.

The Black-Scholes Formula We are given a strike price K , time instance T , interest rate r and volatility σ . Define

$$z = \frac{\log \frac{x}{K} + (r + \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{T - t}}$$

and

$$(16.7) \quad C(x, t) = xN(z) - e^{-r(T-t)}KN(z - \sigma\sqrt{T-t}).$$

We now state the Black Scholes theorem.

Theorem 16.8 (Black Scholes option pricing theorem). *If there is no arbitrage then the price of an option with strike K and maturity time T must be equal to $C(S(0), 0)$. In general, the price of this option at time t must be $C(S(t), t)$.*

Thus, as the theorem claims, the current price of the option is uniquely determined by the current price of the stock and the market parameters. Notice that while the price does depend on the volatility σ of the price process, it does not depend on $\mu = \alpha + \sigma^2/2$, the instantaneous rate of return. This might seem very surprising as presumably an option corresponding to a stock which has a higher rate of return should have higher price. The explanation is that the absence of arbitrage prevents two stocks with the same volatility from having different rates of return. To see this observe the following simple fact.

Lemma 16.9. *Suppose $\sigma = 0$ i.e., the stock is riskless. Then no arbitrage implies $\mu = r$.*

The lemma is pretty much self-evident. If $\mu = \alpha > r$ we can make money by investing in the stock and shortselling the same amount of bonds. Say we buy one dollar worth of stock and sell one dollar worth of bonds. Then at time zero our net investment is zero but at time t the worth of our portfolio is $e^{\alpha t} - e^{rt} > 0$. Similarly, when $\alpha < r$ we can buy bonds and sell stock. Thus it must be (as is obvious) that $\alpha = r$. That is no arbitrage places a constraint on the combinations of the mean rate of return μ and the volatility σ .

Before we discuss how to establish the Black-Scholes formula (we will only sketch the proof) let us discuss its behavior in various "extreme" cases. Say again $\sigma = 0$. In this case it must be that $\alpha = r$. What is the worth of the option at time $t = 0$? Suppose the price of the stock at time zero is x . At time T option pays with probability one an amount $xe^{rT} - K$. If $x^{rT} \geq K$ then its worth at time $t = 0$ is exactly $x - e^{-rT}K$ as we can create a net $xe^{rT} - K$ by investing $x - e^{-rT}K$ into stock or bonds (which are equivalent since $\alpha = r$). Therefore the right price of this option at time zero is exactly $x - e^{-rT}K$. But if $xe^{rT} < K$ or $x - Ke^{-rT} < 0$ then with probability one option does not pay anything. Therefore it is worth zero.

Let us see whether this matches what the Black-Scholes formula predicts. We first compute z as $\sigma \rightarrow 0$. In this case the limit of z is

$$\lim_{\sigma \rightarrow 0} \frac{\log(x/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = \infty$$

when $\log(x/K) + r > 0$ and

$$\lim_{\sigma \rightarrow 0} \frac{\log(x/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = -\infty$$

when $\log(x/K) + r < 0$. The degenerate case $\log(x/K) + r = 0$ is sort of a tie breaking case and we do not interpret it. The two conditions are exactly $x - e^{-rT}K > 0$ or < 0 . In the first case the Black-Scholes formula gives

$$C(x, 0) = xN(\infty) - e^{-rT}KN(\infty) = x - e^{-rT}K.$$

In the second case it gives

$$C(x, 0) = xN(-\infty) - e^{-rT}KN(-\infty) = 0.$$

This is consistent with our findings.

Let us now look at a different approximation as $t \rightarrow T$, but we no longer assume $\sigma = 0$. In this case we have the limit

$$\lim_{t \rightarrow T} \frac{\log(x/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} = \infty$$

when $x > K$ and is $-\infty$ when $x < K$. Also $z - \sigma\sqrt{T - t}$ approaches ∞ when $x > K$ and $-\infty$ when $x < K$. This means that

$$\lim_{t \rightarrow T} C(x, t) = xN(\infty) - KN(\infty) = x - K$$

when $x > K$ and $= 0$ when $x < K$. This is again consistent with common sense. As the strike time T approaches, the uncertainty about the stock gradually disappears and its worth is $x - K$ when $x > K$ and 0 when $x < K$, namely it is worth exactly $\max(x - K, 0)$ - which is its payoff upon maturity.

Proof sketch for the Black-Scholes Theorem. We will show that there exists a self-financed trading strategy $a(t), b(t)$ for trading stocks $S(t)$ and bonds $\beta(t)$ such that $a(T)S(T) + b(T)\beta(T) = \max(S(T) - K, 0)$, that is the value of the portfolio at time T is exactly the payoff $\max(S(T) - K, 0)$ of the option at time T . Since there is no arbitrage then the price of the option at time $t = 0$ must be exactly $a(0)S(0) + b(0)\beta(0)$.

We will first assume that the right price $C(S(t), t)$ of the option at time t is "nice". Specifically the corresponding function $C(x, t)$ is twice continuously differentiable. Later on when we actually find C we simply verify that this is indeed the case. For now, we make this assumption and let us try to infer the function C as well as self-financed strategies a and b .

Using the Ito formula we know that $C(S(t), t)$ is again an Ito process and using (16.6)

$$\begin{aligned} dC &= \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial x} dS + \frac{1}{2} \frac{\partial^2 C}{\partial x^2} (dS)^2 \\ (16.10) \quad &= \left(\frac{\partial C}{\partial t} + \frac{\partial C}{\partial x} \mu S + \frac{1}{2} \frac{\partial^2 C}{\partial x^2} S^2 \sigma^2 \right) dt + \frac{\partial C}{\partial x} S \sigma dB. \end{aligned}$$

We want to find a self-financed trading strategy a, b such that

$$(16.11) \quad a(t)S(t) + b(t)\beta(t) = C(S(t), t).$$

How can we find it? We will find an Ito representation of $a(t)S(t) + b(t)\beta(t)$ and match it with Ito representation of C . From a self-financed condition (16.4) we need to have

$$\begin{aligned} a(t)S(t) &= a(0)S(0) + \int_0^t a(s)dS(s), \\ b(t)\beta(t) &= b(0)\beta(0) + \int_0^t b(s)d\beta(s). \end{aligned}$$

or in differential form

$$(16.12) \quad d(as + b\beta) = adS + bd\beta = (a\mu S + b\beta r)dt + aS\sigma dB.$$

Now we would like to match this with (16.10). Let us set $a = \frac{\partial C}{\partial x}$. This way we match both the dB multipliers and make $a\mu S = \frac{\partial C}{\partial x} \mu S$. Now the equality (16.11) suggest then taking

$$(16.13) \quad b(t) = \frac{1}{\beta(t)}(C(S(t), t) - a(t)S(t)) = \frac{1}{\beta(t)}(C(S(t), t) - \frac{\partial C}{\partial x} S(t))$$

On the other hand we need to match the remaining dt coefficients in (16.12) and (16.10):

$$b\beta r = \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial x^2} S^2 \sigma^2$$

which using (16.13) leads to

$$rC(S(t), t) - r \frac{\partial C}{\partial x} S(t) = \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial x^2} S^2 \sigma^2$$

This defines a partial differential equation on C . To this equation we have a boundary condition $C(x, T) = \max(x - T, 0)$ (as this is the only arbitrage free price of the option at time T). It turns out (and quite miraculously so) that this PDE has indeed an explicit solution of the form (16.7). In retrospect we check that this solution is twice continuously differentiable. \square

In order to make this proof rigorous, we would just take the guessed solution plug it in and check that the implied solution for a and b makes them a self-financed trading strategy. Our proof approach was more in line of how this formula was discovered.

16.3. Additional reading materials

- Duffie "Dynamic Asset Pricing Theory" [1].

BIBLIOGRAPHY

1. D. Duffie, *Dynamic asset pricing theory*, Princeton University Press, 2001.