

## Solution for 18.112 Mid 2

### Problem 1.

*Solution:* The function

$$f(z) = \frac{1}{e^z - 1}$$

is analytic in  $\mathbb{C} - \{2n\pi i, n \in \mathbb{Z}\}$ , and has simple pole at points  $z = 2n\pi i$ . Thus there are three poles in the region bounded by  $\gamma$ , which correspond to  $n = 0, \pm 1$ . Moreover, at each pole  $z$ , the residue equals to

$$\frac{1}{(e^z - 1)'} = \frac{1}{e^z} = 1.$$

By residue theorem,

$$\int_{\gamma} \frac{1}{e^z - 1} dz = 2\pi i(1 + 1 + 1) = 6\pi i.$$

### Problem 2.

*Solution:* Let

$$u(z) = \operatorname{Re} f(z),$$

then by formula (66), there exists constant  $C$  such that for any  $|z| < R$ ,

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{\zeta + z}{\zeta - z} u(\zeta) \frac{d\zeta}{\zeta} + iC.$$

Thus

$$f'(z) = \frac{1}{\pi i} \int_{|\zeta|=R} \frac{u(\zeta)}{(\zeta - z)^2} d\zeta$$

for any  $|z| < R$ .

Now for any  $z$ , take  $R > 2|z|$  large enough such that

$$\frac{|u(\zeta)|}{|\zeta|} < 1$$

for any  $|\zeta| \geq R$ . Then

$$|f'(z)| \leq \frac{1}{\pi} \int_{|\zeta|=R} \frac{|\zeta|}{|\zeta - z|^2} |d\zeta| \leq \frac{1}{\pi} \cdot 2\pi R \cdot \frac{R}{(R/2)^2} = 8.$$

Thus  $f'(z)$  is bounded analytic function on  $\mathbb{C}$ . By Liouville's theorem,  $f'(z)$  is constant, so  $f(z) = az + b$  is linear. By condition

$$\frac{u(z)}{z} \rightarrow 0,$$

we see that  $a = 0$ , which implies that  $f$  is a constant.

**N.B.** You can also prove that

$$\frac{\operatorname{Im}f(z)}{z} \rightarrow 0,$$

thus

$$\frac{f(z)}{z} \rightarrow 0 \text{ as } z \rightarrow \infty.$$

So by *Problem 4* or *Problem 5* in **Mid 1**,  $f$  is a polynomial, and thus  $f$  is a constant.

### Problem 3.

*Solution:* By Cauchy's formula,

$$\begin{aligned} |f^{(n)}(0)| &\leq \frac{n!}{2\pi} \int_{|\zeta|=r} \frac{|f(\zeta)|}{|\zeta^{n+1}|} |d\zeta| \\ &\leq \frac{n!}{2\pi} \cdot 2\pi r \cdot \frac{1}{1-r} \frac{1}{r^{n+1}} \\ &= \frac{n!}{(1-r)r^n} \end{aligned}$$

for  $0 < r < 1$ . On the other hand,

$$\begin{aligned} \frac{1}{(1-r)r^n} &= \frac{1}{n^n} \frac{1}{(1-r)(r/n)^n} \\ &\geq \frac{1}{n^n} \left( \frac{n+1}{1-r + \frac{r}{n} + \cdots + \frac{r}{n}} \right)^{n+1} \\ &= \frac{(n+1)^{n+1}}{n^n} \end{aligned}$$

by Algebraic-Geometric Mean Value inequality, with equality holds if and only if

$$1 - r = \frac{r}{n},$$

i.e.

$$r = \frac{n}{n+1}.$$

So the best estimate of  $f^{(n)}(0)$  that Cauchy's formula will yield is

$$|f^{(n)}(0)| \leq \frac{n!(n+1)^{(n+1)}}{n^n} = (n+1)!(1 + \frac{1}{n})^n.$$

**N.B.** You can also get the minimal value of

$$\frac{1}{(1-r)r^n}$$

by computing derivatives.

**Problem 4.**

*Solution:* Let

$$\begin{aligned} f(z) &= z^7 - 2z^5 + 6z^3 - z + 1, \\ g(z) &= 6z^3, \\ h(z) &= z^7. \end{aligned}$$

Then we have

$$|f(z) - g(z)| = |z^7 - 2z^5 - z + 1| \leq 5 < 6 = |g(z)|$$

on the curve  $|z| = 1$ , and

$$|f(z) - h(z)| = |-2z^5 + 6z^3 - z + 1| \leq 115 < 128 = |h(z)|$$

on the curve  $|z| = 2$ . Thus by Rouché's theorem,  $f$  has 3 roots (as  $g$ ) in  $|z| < 1$  and 7 roots (as  $h$ ) in  $|z| < 2$ .

**N.B.** You can also choose  $g(z)$  to be  $6z^3 + 1$  or  $6z^3 - z$ , or choose  $h(z) = z^7 + 1$  etc.