MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Physics

Physics 8.04 **Spring 2006**

EXAM 2
Tuesday, March 14, 2006

11:00am-12:30pm

FAMILY (Last) NAME

GIVEN (First) NAME

Student ID Number

Instructions:

1. SHOW ALL WORK. All work must be done in the exam booklet.
2. This is a closed book exam.
3. BOOKS, NOTES, COMPUTERS and CELL PHONES are NOT ALLOWED.
4. Do all FOUR (4) problems.
5. You have 90 minutes to solve the problems. Exams will be collected at 12:30pm.

Problem	Maximum	Score	Grader
1	30		
2	30		
3	20		
4	20		
TOTAL	100		

A. Selected formulas

Time-dependent Schroedinger equation: $i\hbar \frac{\partial}{\partial t}\Psi(x,t) = \hat{H}\Psi(x,t)$

Time-independent Schroedinger equation: $\hat{H}\psi(x) = E\psi(x)$

Position representation of Hamiltonian operator $\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\right)^2 + V(x)$

Position representation of momentum operator $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$

Momentum representation of position operator $\hat{x} = i\hbar\frac{\partial}{\partial p}$

Probability current $J(x,t) = \frac{\hbar}{2im}\left(\Psi^*(x,t)\frac{\partial\Psi(x,t)}{\partial x} - \frac{\partial\Psi^*(x,t)}{\partial x}\Psi(x,t)\right)$

Fourier transform $\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x)e^{-ikx}dx$ $\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{ikx}dk$

Heisenberg uncertainty $\Delta x\Delta p \geq \frac{\hbar}{2}$

Uncertainty of a quantity $(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2$

Infinite potential well $V(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq a \\ \infty & \text{elsewhere} \end{cases}$

$E_n = \frac{\hbar^2\pi^2}{2ma^2}n^2$ $u_n(x) = \sqrt{\frac{2}{a}}\sin(n\pi\frac{x}{a})$

Expansion coefficients $c_n = \int_{-\infty}^{\infty}u_n^*(x)\psi(x)dx$

Delta function potential $V(x)=A\delta(x)$ $\psi'(+\varepsilon) - \psi'(-\varepsilon) = \frac{2mA}{\hbar^2}\psi(0)$

Commutator between p and x $[\hat{p},\hat{x}] = \frac{\hbar}{i}$

Hermitian adjoint operator $\int_{-\infty}^{\infty}dx\left(\hat{A}^+\psi_1(x)\right)^*\psi_2(x) = \int_{-\infty}^{\infty}dx\psi_1^*(x)\hat{A}\psi_2(x)$

Please note the additional formulas compared to the practice exam.

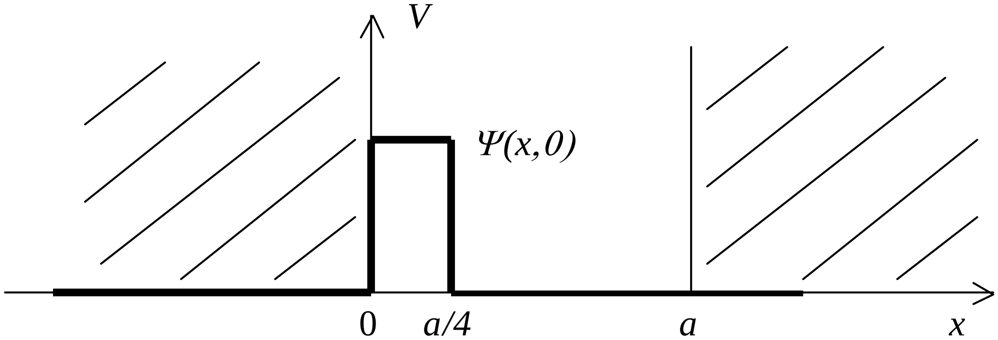
1. Time evolution in infinite well. (30 points)

A particle of mass m is in a one-dimensional infinite potential well of width a given by the potential

$$V(x) = \begin{cases} 0 & \text{for } 0 \le x \le a \\ \infty & \text{elsewhere} \end{cases}.$$

Suppose that at time t=0 the particle is located in the leftmost quarter of the box,

$$\Psi(x, t = 0) = \begin{cases} \sqrt{4/a} & \text{for } 0 \le x \le a/4 \\ 0 & \text{elsewhere} \end{cases}.$$



The eigenenergies and eigenfunction for this potential are given in the formula part of this exam.

a) (10 points) Write the expansion of the wave function $\Psi(x,0)$ in terms of energy eigenfunctions and explicitly compute the expansion coefficients c_n.

b) (5 points) Give an expression for $\Psi(x,t)$ at arbitrary later time t.

c) (5 points) If an energy measurement is made, what values will be observed, and with what probabilities? Are there energy eigenvalues that will never be observed?

d) (3 points) Do you expect $\langle p \rangle$ to change in time and why or why not?

e) (7 points) Give an expression for the expectation value of the inverse of energy, $\left\langle \frac{1}{E} \right\rangle$, as a function of time.

You might find the following formulas useful:

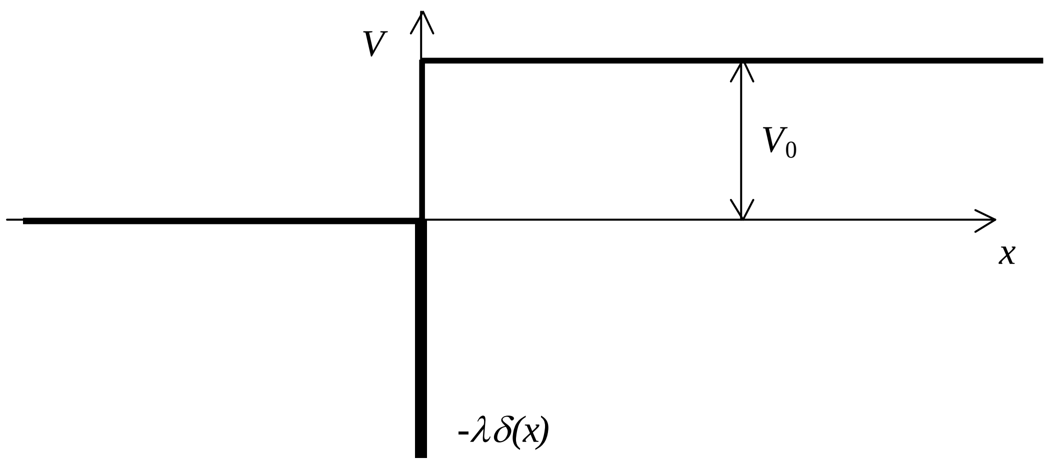
$$\sin^2\left(\frac{y}{2}\right) = \frac{1}{2}(1 - \cos y) \qquad \frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^4}\sin^4\left(\frac{n\pi}{8}\right) = 0.167$$

2. Bound-state problem with delta function and potential step. (30 points)

An attractive delta-function well located at the origin is superimposed with a potential step at the origin,

$V(x) = -\lambda\delta(x) + V_0\Theta(x)$,

where $\Theta(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$ is the step function, and λ, V_0 are positive quantities.



a) (5 points) Write down the wavefunctions in the regions $x>0$ and $x<0$ for a bound-state problem. Is the condition for a bound state $E<0$ or $E<V_0$, and why?

b) (3 points) What are the units of the constant λ?

c) (7 points) Write down the wavefunction matching conditions at $x=0$.

d) (5 points) Without calculation, but using knowledge of the wavefunctions, decide if for the bound state we should find $\langle x\rangle<0$, $\langle x\rangle=0$, or $\langle x\rangle>0$.

e) (10 points) Find an equation that determines the energy of the bound state in terms of fundamental constants and λ, V_0, *m*.

3. Operators and commutation relations (20 points)

An operator $\hat{A}$ is defined as $\hat{A} = a\hat{x} + ib\hat{p}$, where a,b are real numbers.

a) (8 points) What is the Hermitian adjoint operator $\hat{A}^{+}$?

b) (12 points) Calculate the commutators $[\hat{A}, \hat{A}]$, $[\hat{A}, \hat{x}]$ and $[\hat{A}, \hat{p}]$.

4. **Exponential factor applied to momentum wavefunction** (20 points).

Consider a one-dimensional wavefunction in momentum space $\phi(p)$, such that $\langle x \rangle = x_0$, and $\langle p \rangle = p_0$, where x_0 and p_0 are constants. Define a new momentum space wavefunction $\phi_1(p) = \phi(p)\exp(ipx_1/\hbar)$, where x_1 is real.

a) (5 points) What is the expectation value $\langle p \rangle$ for the wavefunction ϕ_1 in terms of the quantities given above?

b) (10 points) What is the expectation value $\langle x \rangle$ for the wavefunction ϕ_1 in terms of the quantities given above?

c) (5 points) Based on your results for a) and b), interpret in one or two sentences the physical significance of the factor $\exp(ipx_1/\hbar)$ applied to a wavefunction in momentum space.