WATER WAVES

BASIC FLUID MECHANICS

(a) CONSERVATION OF MASS

$$\rho = \text{const} \quad \Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

(b) NEWTON'S 2ND LAW

$$\rho = \text{const}$$

$$\rho\left\{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right\}$$

$$= -\frac{\partial p}{\partial x} + \mu\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right\}$$

other equations by rotational symmetry

$$\begin{tikzcd} & u \arrow[dr] & \\ w \arrow[ur] & & v \arrow[ll] \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr] & \\ z \arrow[ur] & & y \arrow[ll] \end{tikzcd}$$

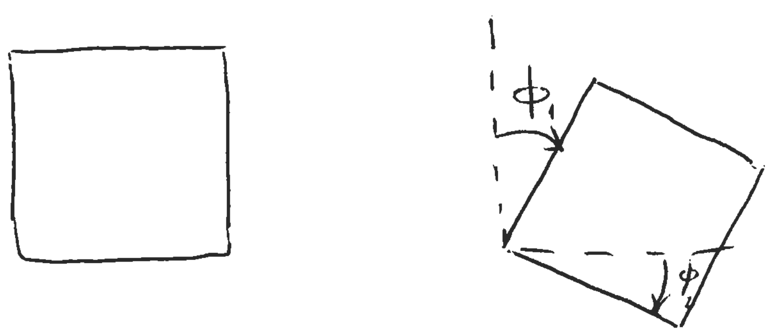
except if gravity prestent, there is

Rotation in fluids

Unlike in rigid bodies, rotation is defined by the sum of rotation of two perpendicular axes (divided by two)

$$\phi = \frac{\phi_1 - \phi_2}{2}$$

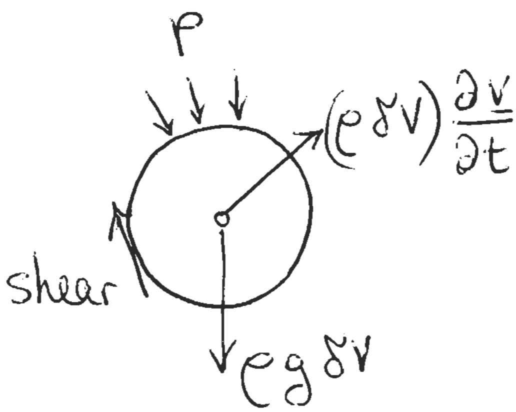
original $\longrightarrow$ final



$$\phi = \frac{\phi_1 + \phi_2}{2}$$

$$\underline{\omega} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)^T$$

Spherical particle



The only force that causes rotation is the shear force

Reynolds Number

$$Re = \frac{Ud}{\nu} = \left\langle \frac{\text{INERTIA}}{\text{Viscous}} \right\rangle$$

$Re \gg 1$ for ocean engineering flows

$\Rightarrow$ Rotation is absent except near walls, bodies, etc, and their wakes

IRROTATIONAL FLOW

$$\underline{\omega} = 0$$

$$\Rightarrow \underline{V} = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$$

$$= \nabla \phi$$

BASIC EQUATIONS ($\rho = \text{const}$)

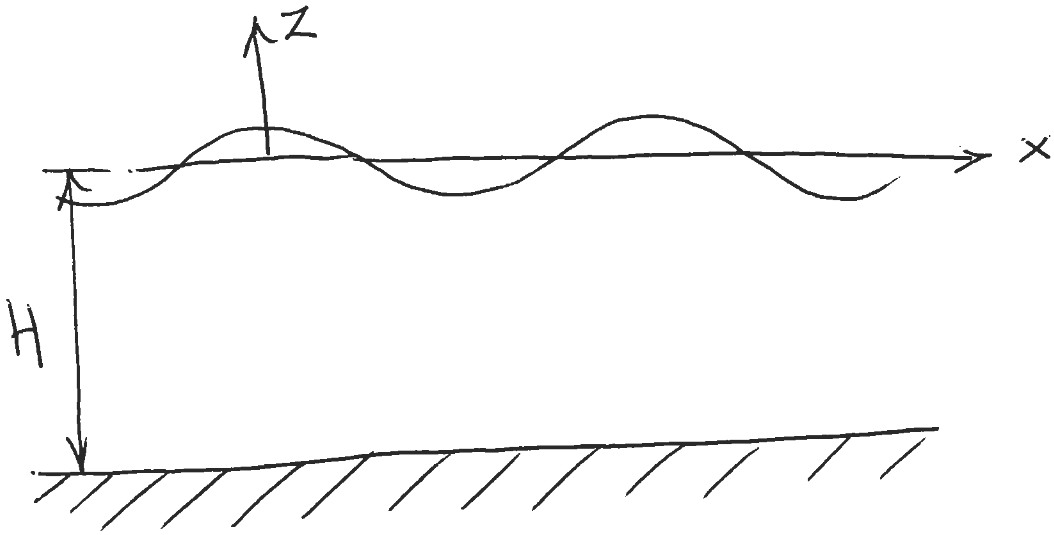
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

conservation of mass

$$p + \rho \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho V^2 + \rho g z = \text{const}$$

Bernoulli (integrated Newton's law)

WATER WAVES



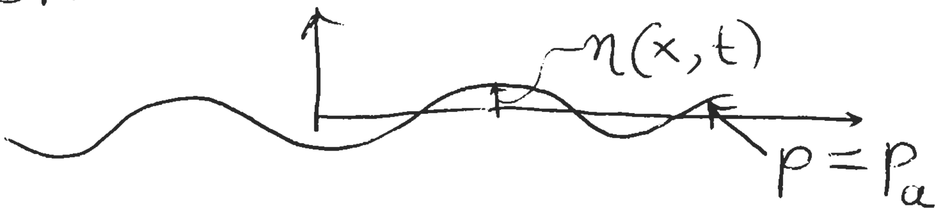
Observation: When height / wavelength $\sim \frac{1}{7}$ wave breaks

$$\Rightarrow \boxed{\frac{\text{amplitude}}{\text{wavelength}} \ll 1}$$

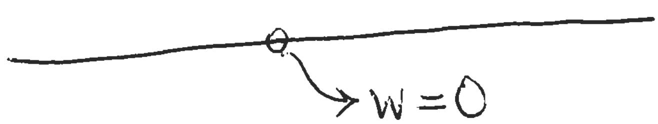
(typically $\frac{1}{50}$ to $\frac{1}{20}$)

LINEAR WAVES DESCRIBE STORMS QUITE WELL (except extreme cases)

SMALL AMPLITUDE WAVES



$$\nabla^2 \phi = 0$$



$$p = p_a \Rightarrow \rho \frac{\partial \phi}{\partial t} + \frac{1}{2}\rho V^2 + \rho g \eta = 0 \quad \text{at } z = \eta$$

$$\boxed{\rho \frac{\partial \phi}{\partial t} + \rho g \eta = 0 \quad \text{at } z = 0}$$

a particle at the surface always on it

$$\frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y} = w \quad \text{at } z = \eta$$

$$\boxed{\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{at } z = 0}$$

Combine free surface conditions

$$\boxed{\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0 \qquad \text{on } z=0}$$

<u>A solution</u>

$$\eta(x,t) = a\cos(\overbrace{\omega t - kx + \psi}^{\theta})$$

$$\phi(x,z,t) = -\frac{a\omega}{k} f(z)\sin\theta$$

$$f(z) = \frac{\cosh[k(z+H)]}{\sinh(kH)}$$

where $\omega = \frac{2\pi}{T}$, $k = \frac{2\pi}{\lambda}$

$$\omega^2 = kg\tanh(kH)$$

$$0 \le \psi < 2\pi$$

$$u(x,z,t) = a\omega\, f(z)\, \cos\theta$$

$$w(x,z,t) = -a\omega\, \frac{1}{k}\, \frac{df(z)}{dz}\, \sin\theta$$

$$p \cong -\rho \frac{\partial \phi}{\partial t} - \rho g z$$

$$= \underbrace{\rho \frac{a\omega^2}{k} f(z) \cos\theta}_{\substack{\text{dynamic} \\ p_d}} - \underbrace{\rho g z}_{\text{hydrostatic}}$$

$$p_d = \rho g\, \eta(x,t)\, \frac{\cosh[k(z+H)]}{\cosh(kH)}$$

Motion of fluid particles is an ellipse

$$\xi_p(x,z,t) = a\, f(z)\, \sin\theta$$

$$\eta_p(x,z,t) = a\, \frac{1}{k}\, \frac{df(z)}{dz}\, \cos\theta$$

Deep water

$$\eta(x,t) = a\cos(\omega t - kx + \phi)$$

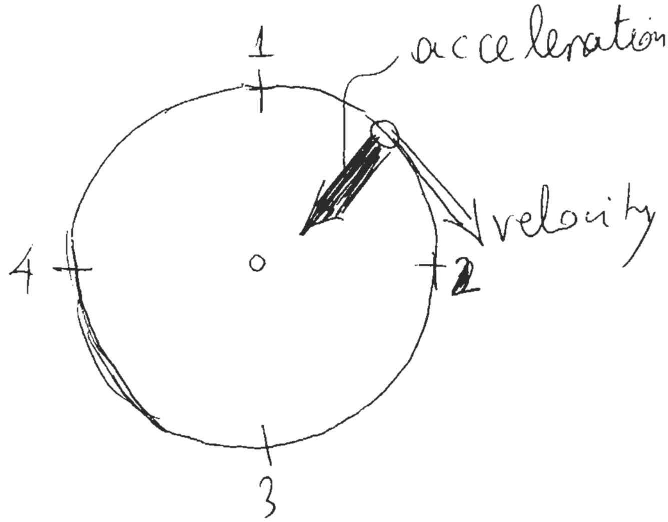
$$\phi(x,z,t) = -\frac{a\omega}{k} e^{kz} \sin\theta$$

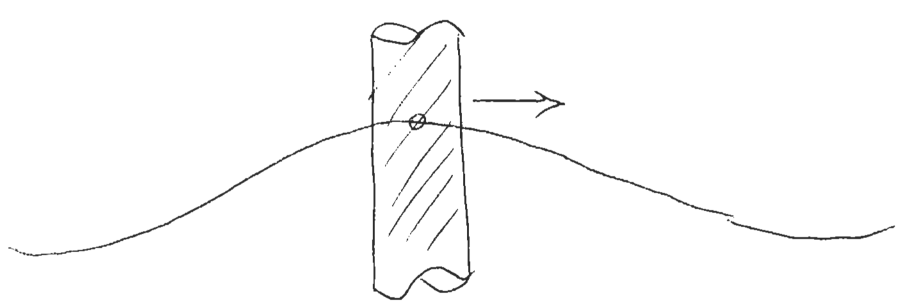
$$p_d = \rho g a e^{kz} \cos\theta$$

Fluid particles move on circles with diameter $2ae^{kz}$

$$\xi_p = a e^{kz} \sin\theta \ , \quad \eta_p = a e^{kz} \cos\theta$$

$$u = a\omega e^{kz} \cos\theta \ , \quad w = -a\omega e^{kz} \sin\theta$$





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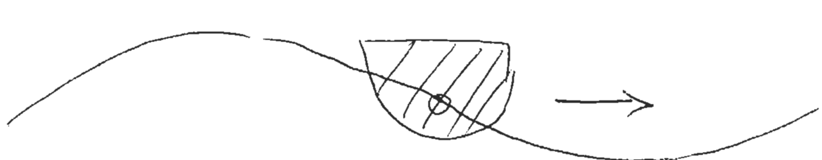

INERTIA