

**PROBLEM SET 4**  
**Solutions**

**Problem 1**

Step responses A and D clearly exhibit a “long tail” due to a combination of a slow, real-valued pole and a nearby zero, so they must correspond to transfer functions (3) and (4). To tell which is which, we need to find the sign of the residue for the real-valued pole in the step responses for (3) and (4). In (3), there will be a positive vector from the zero combined with a negative vector from the step input pole at  $s = 0$ , so the residue will be negative. That implies that the mode due to the slow pole will be subtracted from the steady-state value, and so the output will approach its steady-state value from below. Therefore, step response D corresponds to transfer function (3). Similarly, the residue for the real-valued pole in the step response for (4) will be positive, so the output will approach steady-state from above, so step response A must correspond to transfer function (4).

The other two step responses are easy: as the zero moves towards the right, peak time decreases and P.O. increases. Therefore B corresponds to (2), and C corresponds to (1).

**Problem 2**

1.

$$\begin{aligned} |A| = -1 &\Rightarrow A^{-1} = \frac{1}{-1} \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \\ |B| = 2 &\Rightarrow B^{-1} = \frac{1}{2} \begin{bmatrix} 8 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1.5 \\ -1 & 0.5 \end{bmatrix} \\ |C| = 0 &\Rightarrow C \text{ is singular, so } C^{-1} \text{ does not exist.} \end{aligned}$$

2.

$$\begin{aligned} |A| &= (1) \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} - (4) \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + (0) \begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix} \\ &= (1)(-2) - (4)(-3) \\ &= 10 \end{aligned}$$

Since  $|A| \neq 0$ , the matrix must have full rank, so  $A$  has rank = 3.

$$\begin{aligned} |B| &= (1) \begin{vmatrix} 4 & 0 \\ -8 & 18 \end{vmatrix} - (2) \begin{vmatrix} -1 & 0 \\ 5 & 18 \end{vmatrix} + (9) \begin{vmatrix} -1 & 4 \\ 5 & -8 \end{vmatrix} \\ &= (1)(72) - (2)(-18) + (9)(-12) \\ &= 0 \end{aligned}$$

Since  $|B| = 0$ , the matrix must not have full rank. By inspection, the first two rows of  $B$  are linearly independent, so the third row must be a linear combination of the first two rows (in fact, the third row is equal to 2 times the first row minus 3 times the second row). So  $B$  has rank = 2.

$A$  is non-singular, but  $B$  is singular because  $|B| = 0$ .  $A$  would be a valid transformation matrix for a 3-element state vector, but  $B$  would not, because of its singularity.

3. (a)

$$\begin{aligned} \det(\lambda I - A) = 0 &\Rightarrow \begin{vmatrix} \lambda - 1 & -3 \\ 0 & \lambda - 2 \end{vmatrix} = 0 \\ &\Rightarrow (\lambda - 1)(\lambda - 2) - (-3)(0) = 0 \\ &\Rightarrow \lambda^2 - 3\lambda + 2 = 0 \\ &\Rightarrow \lambda_1 = 1, \lambda_2 = 2 \\ \lambda = \lambda_1 = 1 &\Rightarrow (\lambda_1 I - A)\vec{v}_1 = 0 \\ &\Rightarrow \begin{bmatrix} 0 & -3 \\ 0 & -1 \end{bmatrix} \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \lambda = \lambda_2 = 2 &\Rightarrow (\lambda_2 I - A)\vec{v}_2 = 0 \\ &\Rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\Rightarrow \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \end{aligned}$$

Similarly for  $B$ :

$$\begin{aligned} \det(\lambda I - B) = 0 &\Rightarrow \lambda^2 - 7\lambda + 12 = 0 \Rightarrow \lambda_1 = 3, \lambda_2 = 4 \\ \lambda = \lambda_1 = 3 &\Rightarrow \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \lambda = \lambda_2 = 4 &\Rightarrow \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{aligned}$$

- (b) i.  $B\vec{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$  so the vector  $\vec{x}$  has been rotated from an angle of  $63^\circ$  to an angle of  $79^\circ$ , and its length has been scaled from  $\sqrt{5}$  to  $\sqrt{26}$ .
- ii. If  $\vec{v} = \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , then  $B\vec{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\vec{v}$ . So the vector  $\vec{v}$  has not been rotated, but its length has been scaled by a factor of 3, which is equal to the eigenvalue associated with that eigenvector.

### Problem 3

1. Define  $x_1 = y$ ,  $x_2 = \dot{y}$ , and  $x_3 = \ddot{y}$ . So we have the state equations:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -2x_1 - 3x_2 + 2x_3 + u \end{aligned}$$

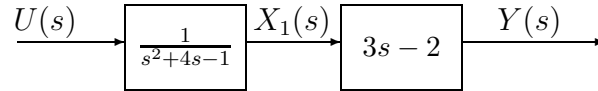
And the output equation:

$$y = x_1$$

The state-space model is then:

$$\begin{aligned}\dot{\vec{x}} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= [1 \ 0 \ 0] \vec{x} + [0] u\end{aligned}$$

2. Split up the system into poles and zeros as follows:



From the first block, we have:

$$\ddot{x}_1 + 4\dot{x}_1 - x_1 = u$$

Let  $x_1$  be the first state variable, and  $x_2 = \dot{x}_1$  be the second state variable. Then the state equation is:

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 1 & -4 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Looking at the second block, we have:

$$y = 3\dot{x}_1 - 2x_1 = 3x_2 - 2x_1$$

And so the output equation is:

$$y = [-2 \ 3] \vec{x} + [0] u$$

#### Problem 4

Use the formula:

$$G(s) = C(sI - A)^{-1}B + D$$

(a)

$$\begin{aligned}sI - A &= \begin{bmatrix} s & 1 \\ 0 & s+2 \end{bmatrix} \\ \Rightarrow (sI - A)^{-1} &= \frac{1}{s(s+2)} \begin{bmatrix} s+2 & -1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{-1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix} \\ \Rightarrow G(s) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & \frac{-1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{s(s+2)} \\ \frac{1}{s+2} \end{bmatrix}\end{aligned}$$

(b)

$$\begin{aligned} sI - A &= \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \\ \Rightarrow (sI - A)^{-1} &= \frac{1}{s(s+3)+2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \\ \Rightarrow G(s) &= \frac{1}{(s+1)(s+2)} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \frac{1}{(s+1)(s+2)} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & s+3 \\ s & -2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \\ -\frac{s-1}{(s+1)(s+2)} & \frac{s+5}{(s+1)(s+2)} \end{bmatrix} \end{aligned}$$

### Problem 5

Define  $d_1$  as the position of  $m_1$ ,  $d_2$  as the position of  $m_2$ , and  $d_3$  as the position of  $m_3$ , where  $d_1$ ,  $d_2$ , and  $d_3$  are measured from the left of the diagram (the same way as  $y$  is measured). Then write down the mass-spring-damper equation for each mass:

$$\begin{aligned} m_1 \ddot{d}_1 &= F - c_1(\dot{d}_1 - \dot{d}_2) - k_1(d_1 - d_2) \\ m_2 \ddot{d}_2 &= -c_2(\dot{d}_2 - \dot{d}_3) - k_2(d_2 - d_3) + c_1(\dot{d}_1 - \dot{d}_2) + k_1(d_1 - d_2) \\ m_3 \ddot{d}_3 &= c_2(\dot{d}_2 - \dot{d}_3) + k_2(d_2 - d_3) \end{aligned}$$

Now define our states as:  $x_1 = d_1$ ,  $x_2 = d_2$ ,  $x_3 = d_3$ ,  $x_4 = \dot{d}_1$ ,  $x_5 = \dot{d}_2$ ,  $x_6 = \dot{d}_3$ . Also define the input as  $u = F$ . So the above equations can be rewritten as:

$$\begin{aligned} \dot{x}_4 &= \frac{1}{m_1}(-k_1x_1 + k_1x_2 - c_1x_4 + c_1x_5 + u) \\ \dot{x}_5 &= \frac{1}{m_2}(k_1x_1 - (k_1 + k_2)x_2 + k_2x_3 + c_1x_4 - (c_1 + c_2)x_5 + c_2x_6) \\ \dot{x}_6 &= \frac{1}{m_3}(k_2x_2 - k_2x_3 + c_2x_5 - c_2x_6) \end{aligned}$$

In addition, from the way we defined our states, we have:

$$\begin{aligned} \dot{x}_1 &= x_4 \\ \dot{x}_2 &= x_5 \\ \dot{x}_3 &= x_6 \end{aligned}$$

And finally, the output equation is:

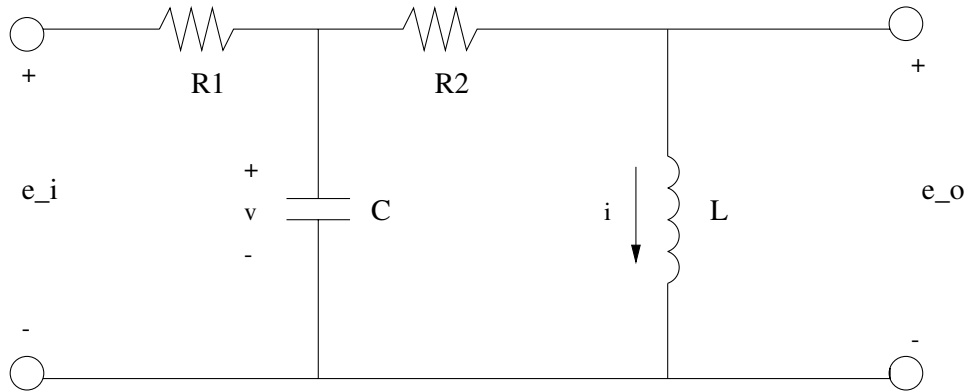
$$y = x_3$$

So now we can write the state-space model:

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & \frac{k_1}{m_1} & 0 & -\frac{c_1}{m_1} & \frac{c_1}{m_1} & 0 \\ \frac{k_1}{m_2} & -\frac{k_1+k_2}{m_2} & \frac{k_2}{m_2} & \frac{c_1}{m_2} & -\frac{c_1+c_2}{m_2} & \frac{c_2}{m_2} \\ 0 & \frac{k_2}{m_3} & -\frac{k_2}{m_3} & 0 & \frac{c_2}{m_3} & -\frac{c_2}{m_3} \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [0 \ 0 \ 1 \ 0 \ 0 \ 0] \vec{x} + [0] u$$

### Problem 6



Define our states to be  $x_1 = v$ , the voltage across the capacitor, and  $x_2 = i$ , the current through the inductor. The input is  $u = e_i$  and the output is  $y = e_o$ . Applying KCL at the node between  $R_1$  and  $R_2$  gives:

$$\frac{v - e_i}{R_1} + C \frac{dv}{dt} + i = 0 \quad \Rightarrow \quad \frac{dv}{dt} = -\frac{1}{R_1 C} v - \frac{1}{C} i + \frac{1}{R_1 C} e_i$$

Applying KVL around the loop containing  $C$ ,  $R_2$ , and  $L$ , gives:

$$v - R_2 i - L \frac{di}{dt} = 0 \quad \Rightarrow \quad \frac{di}{dt} = \frac{1}{L} v - \frac{R_2}{L} i$$

Finally, the output voltage  $e_o$  is given by:

$$e_o = v - R_2 i$$

So the state-space model is:

$$\dot{\vec{x}} = \begin{bmatrix} -\frac{1}{R_1 C} & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R_2}{L} \end{bmatrix} \vec{x} + \begin{bmatrix} \frac{1}{R_1 C} \\ 0 \end{bmatrix} u$$

$$y = [1 \quad -R_2] \vec{x} + [0] u$$