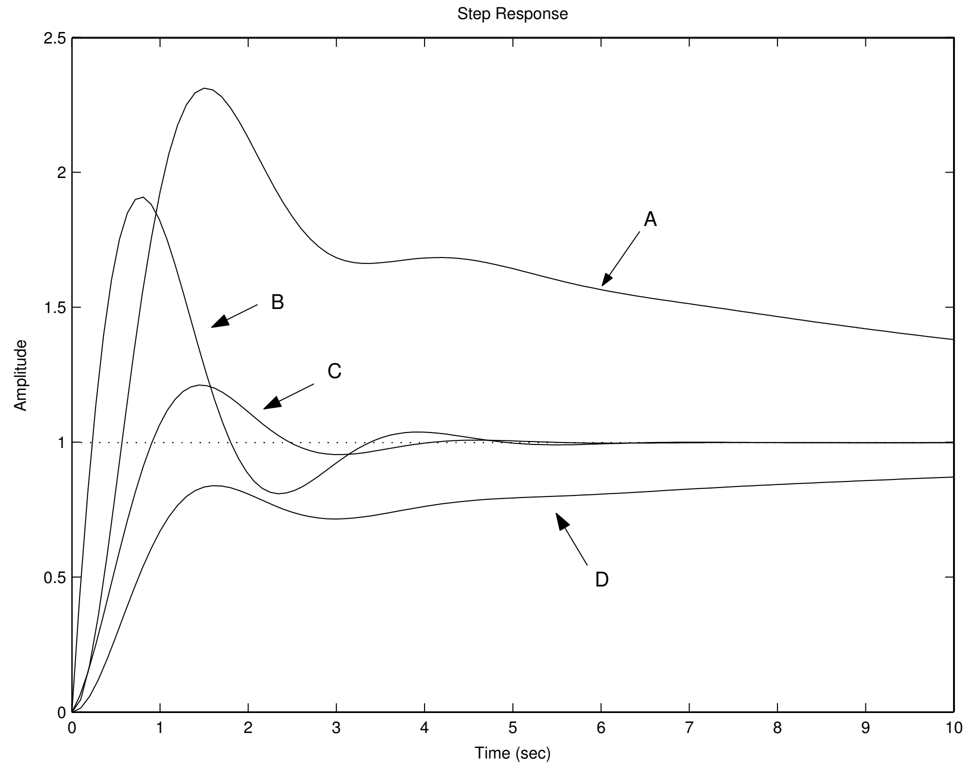
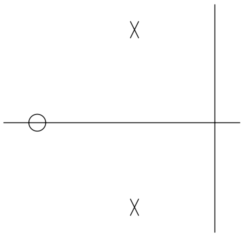
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Aeronautics and Astronautics
16.060: Principles of Automatic Control
Fall 2003

PROBLEM SET 4
Due: 10/9/2003

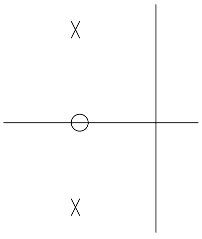
Problem 1: Effects of Zeros

The first figure below shows four different step responses, and the second figure shows the pole-zero plots for four different transfer functions. For each step response, pick the pole-zero plot whose transfer function best matches the response, and explain your decision. Note that the complex poles in (1)-(4) are in the same location, and the real-valued pole in (3) and (4) is in the same location.

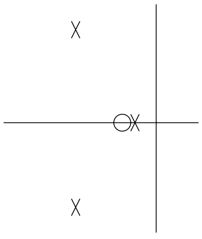




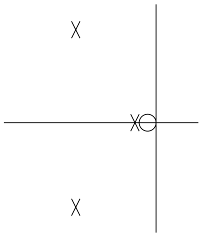
(1)



(2)



(3)



(4)

Problem 2: Matrix Algebra Review

1. Calculate the inverse of each of the following matrices. If the inverse does not exist, explain why.

   $$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$$

2. Calculate the determinant and find the rank of each of the following matrices. Also specify whether or not each matrix is singular. Which of these matrices would be a valid transformation matrix for a state vector?

   $$\mathbf{A} = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 & 9 \\ -1 & 4 & 0 \\ 5 & -8 & 18 \end{bmatrix}$$

3. (a) Calculate the eigenvalues and eigenvectors of each of the following matrices.

   $$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}$$

   (b) Recall that a matrix can be thought of as a linear transformation that "transforms" one vector into another. A given matrix will transform most vectors by (1) rotating them, and (2) scaling them (i.e. changing their lengths). However, for a given matrix, there are certain vectors that will get scaled but not rotated when you multiply them by the matrix. These special vectors are called the *eigenvectors*, and the associated scaling factors are called the *eigenvalues.* If $\mathbf{A}$ is a matrix, and $\vec{v}$ is an eigenvector of $\mathbf{A}$, and λ is the associated eigenvalue, then $\mathbf{A}\vec{v} = \lambda\vec{v}$. In other words, if you multiply a matrix and one of its eigenvectors, it's the same as multiplying a constant (equal to the eigenvalue) and the eigenvector.

      i. Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. For the matrix $\mathbf{B}$ from part (a), calculate $\mathbf{B}\vec{x}$. Sketch the vectors $\vec{x}$ and $\mathbf{B}\vec{x}$. Notice how the matrix $\mathbf{B}$ has rotated and scaled the vector $\vec{x}$.

      ii. Now let $\vec{v}$ be one of the eigenvectors you found for $\mathbf{B}$ (you choose which one). Sketch $\vec{v}$ and $\mathbf{B}\vec{v}$ on the same plot. Did the vector get rotated this time? By what factor did its length get scaled? What was the eigenvalue associated with that eigenvector?

Problem 3: Differential Equation → State-Space Model

Derive the state-space model (i.e. find the A,B,C, and D matrices) for each of the following differential equations. (You can assume zero initial conditions.) Take $u(t)$ to be the input and $y(t)$ to be the output.

1. $\dddot{y} - 2\ddot{y} + 3\dot{y} + 2y = u$
2. $\ddot{y} + 4\dot{y} - y = 3\dot{u} - 2u$

Problem 4: State-Space Model → Transfer Function

Do Problem 11.18 in Van De Vegte.

Problem 5: Deriving a State-Space Model for a Mechanical System

Do Problem 11.13 in Van De Vegte. Take the force F to be the input.

Problem 6: Deriving a State-Space Model for an Electrical System

Derive the state-space model for the following RLC circuit. Take the voltage e_i to be the input and the voltage e_o to be the output. R_1, R_2, C, and L are known constants.

Hint: Use the constitutive equations:

- Resistor: $V = RI$
- Capacitor: $I = C\frac{dV}{dt}$
- Inductor: $V = L\frac{dI}{dt}$

Also, recall that state variables typically include the voltage across a capacitor and the current through an inductor.