16.06 Lecture 26

Frequency Response Analysis

November 5, 2003

Today's Topics:

1. Steady state system responses to sinusoidal inputs
2. Second order system example

Consider the following experiment in which a stable linear system is driven by a sinusoidal input

The Laplace Transform of this input is-

If $G(s)$ is the system transfer function then the Laplace Transform of the output is which can be written as-

Where $N(s)$ and $D(s)$ are the numerator and denominator polynomials of $G(s)$.

Then

Where $-z_1, -z_2, ...$ are the system zeros and $-p_1, -p_2, ...$ are the system poles

A partial fraction expansion of $C(s)$ obtains

The inverse Laplace Transform would then be

Since the system is stable all of the system poles lie in the left half plan. Hence all of the $p's$ have positive real parts so each of the terms of the form $e^{-p_i t}$ will decay to zero as time increases. In particular, if we wait for a long time after startup then

This is the steady state response to the sinusoidal input at the frequency ω.

The two residuals K_1 and K_2 can be evaluated in the usual fashion. Recall that

so

and the steady state response is

Now let's look at the term $G(j\omega)$. It maps the point $j\omega$ in the "s" plane to the point $G(j\omega)$ in the "G" plane.

The complex number $G(j\omega)$ can be represented either as the sum of real and imaginary parts-

or as a vector of magnitude $M(\omega)$ at the angle $\phi(\omega)$

where

Also, $G(-j\omega)$ is the complex conjugate of $G(j\omega)$

Using these results we can substitute back into the equation for the steady state response

So what do we have?

<u>Frequency Response</u>: A steady sinusoidal input, of frequency ω and magnitude A, into a stable linear system, yields, after all transients have died out, a steady sinusoidal output at frequency ω, of magnitude $A \cdot M(\omega)$ and with phase shift $\phi(\omega)$. The amplitude ratio $M(\omega)$ and phase shift $\phi(\omega)$ satisfy the expressions-

Said in another way, the input sinusoid has its amplitude multiplied by $M(\omega)$ and its phase shifted by the angle $\phi(\omega)$, as it passes through the system $G(s)$

Second Order System Example-

The transfer function of a second order system, with undamped natural frequency ω_n and damping ratio δ, is

so

which yields the amplitude ratio

and phase shift

At low frequencies, as ω approaches zero

At high frequencies, as ω approaches infinity

At $\omega \quad \omega_n$

**********Freq Rsp plots********