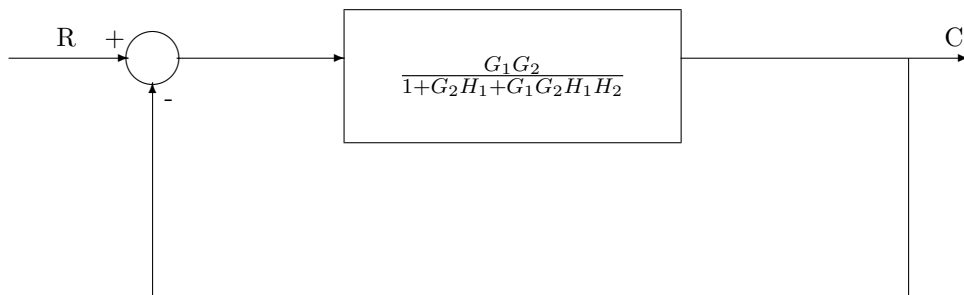
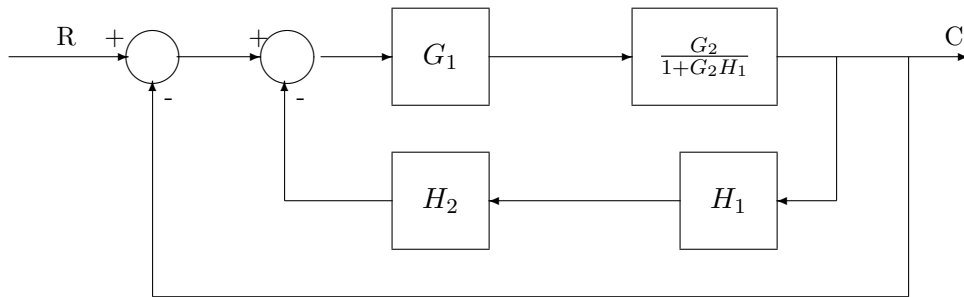
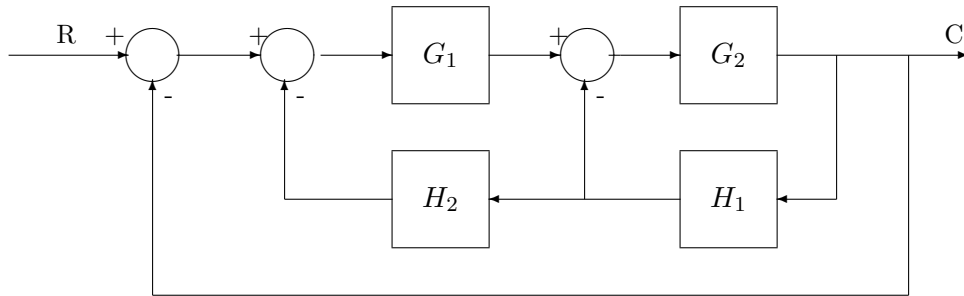


PROBLEM SET 2
Solutions

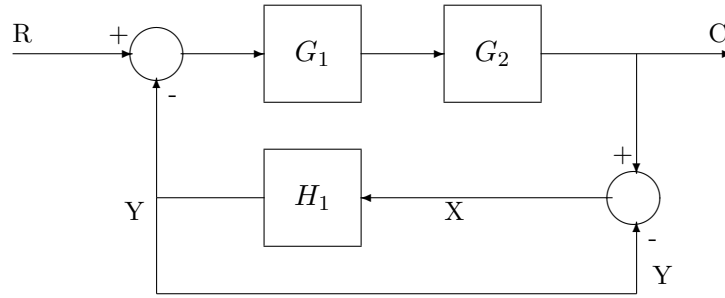
Problem 1

P3.27(a):



$$\Rightarrow T(s) = \frac{G_1 G_2}{1 + G_1 G_2 + G_2 H_1 + G_1 G_2 H_1 H_2}$$

P3.27(b):



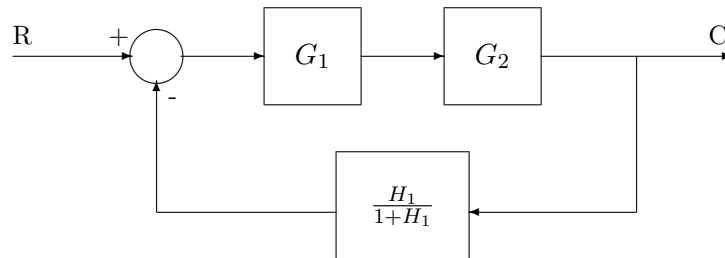
This one is a little trickier. Label the input to H_1 as X , and the output of H_1 as Y , as shown. Therefore, we have:

$$\begin{aligned} X &= C - Y \\ Y &= H_1 X \end{aligned}$$

Solve for X and Y to get:

$$\begin{aligned} X &= \frac{C}{1 + H_1} \\ Y &= \frac{H_1}{1 + H_1} C \end{aligned}$$

So the block diagram can be rewritten as:



And now we can write down the closed-loop transfer function:

$$T(s) = \frac{G_1 G_2 (1 + H_1)}{1 + H_1 (1 + G_1 G_2)}$$

Problem 2

- $f(t)$ is the input force to a standard spring-mass-damper system, so the equation of motion is:

$$m\ddot{x} + c\dot{x} + kx^{2/3} = f \tag{1}$$

- We need to linearize the non-linear spring force $f_k(x)$ about the point $x = x_0$. First, write down the Taylor Series expansion for $f_k(x)$ about $x = x_0$:

$$f_k(x - x_0) = f_k(x_0) + f'_k(x_0)(x - x_0) + \frac{1}{2!}f''_k(x_0)(x - x_0)^2 + \dots \quad (2)$$

$$= kx_0^{2/3} + \frac{2}{3}kx_0^{-1/3}(x - x_0) - \frac{1}{9}kx_0^{-4/3}(x - x_0)^2 + \dots \quad (3)$$

Now we drop all the higher order terms so we're left with only two terms: a constant term and a linear term.

$$f_k(x - x_0) \approx kx_0^{2/3} + \frac{2}{3}kx_0^{-1/3}(x - x_0) \quad (4)$$

Next, we rewrite the original differential equation in terms of deviations from the operating point. Define:

$$\begin{aligned} \hat{x} &= x - x_0 \\ \hat{f} &= f - f_0 \end{aligned}$$

In this case, f_0 represents the input force that will keep the system in equilibrium at the operating point $x = x_0$. To find this force, go back to the original differential equation and set $x = x_0$ and \ddot{x} and \dot{x} to zero (since the mass is in equilibrium):

$$f_0 = kx_0^{2/3} \quad (5)$$

Also, from the way we defined \hat{x} , notice that:

$$\begin{aligned} \dot{\hat{x}} &= \dot{x} \\ \ddot{\hat{x}} &= \ddot{x} \end{aligned}$$

Now substitute everything into the original differential equation:

$$m\ddot{\hat{x}} + c\dot{\hat{x}} + kx_0^{2/3} + \frac{2}{3}kx_0^{-1/3}(x - x_0) = f_0 + \hat{f} \quad (6)$$

And, since $\hat{x} = x - x_0$, and $f_0 = kx_0^{2/3}$, we can write down the final version of our linearized equation for the spring:

$$m\ddot{\hat{x}} + c\dot{\hat{x}} + \frac{2}{3}kx_0^{-1/3}\hat{x} = \hat{f} \quad (7)$$

3. Take the Laplace transform of the linearized differential equation:

$$ms^2\hat{X} + cs\hat{X} + \frac{2}{3}kx_0^{-1/3}\hat{X} = \hat{F} \quad (8)$$

Now rearrange to get the transfer function from the input \hat{F} to the output \hat{X} :

$$\frac{\hat{X}}{\hat{F}} = \frac{1}{ms^2 + cs + \frac{2}{3}kx_0^{-1/3}} \quad (9)$$

- The main limitation of this linearized model is that it is only valid for small deviations from the operating point. To understand why this limitation exists, look back at the Taylor Series expansion for the spring force. The infinite Taylor Series expansion gives the exact value of $f_k(x)$, but when we linearize, we drop the $(x - x_0)^2$ term, the $(x - x_0)^3$ term, etc. As long as x is close to x_0 , these terms are small enough not to worry about, but the further x gets away from the operating point x_0 , the larger these errors will be, and the more useless our linearized model will become.

Problem 3

- First, multiply all the terms in the forward path transfer function and simplify to get:

$$G(s) = K \frac{s^2(s+1)}{s^2(s+2)^2(s+3)} = K \frac{s+1}{s^3+7s^2+16s+12} \quad (10)$$

Then factor out the 12 from the denominator so that the constant coefficients of the polynomials are both one:

$$G(s) = \frac{K}{12} \frac{s+1}{\frac{1}{12}s^3 + \frac{7}{12}s^2 + \frac{4}{3}s + 1} \quad (11)$$

In this system, there are no integrators so the type number of the system is 0. The gain is $\frac{K}{12}$.

- From the steady-state error table, the error for a unit step input will be:

$$e_{ss} = \frac{1}{1 + \frac{K}{12}} = \frac{12}{12 + K} \quad (12)$$

- In a type 1 system, there is an integrator between the error and the output, so the error is the derivative of the output. For a ramp input, there will be a ramp output as well, because that is the only type of output that is consistent with the system in steady-state. (For example, check what happens if you thought the output might be a constant: the error would be the difference between a ramp input and a constant, which is a ramp. But in a type 1 system in steady-state, the output must be the integral of a ramp, which is not a constant \Rightarrow the output cannot be a constant.)

Having established that (1) the output is also a ramp, (2) the error is the derivative of the output, and (3) the derivative of a ramp is a constant, we can infer that the steady-state error must be a constant.

Problem 4

Part 1:

- Use the final value theorem:

$$\omega_{ss} = \lim_{s \rightarrow 0} s\Omega(s) = \lim_{s \rightarrow 0} sR(s)G(s) = \lim_{s \rightarrow 0} s \cdot \frac{1}{s} \cdot \frac{K_t/R_f B}{T_m s + 1} = \frac{K_t}{R_f B} \quad (13)$$

-

$$\begin{aligned} K_t &= 1 \Rightarrow \omega_{ss} = 1 \\ K_t &= 2 \Rightarrow \omega_{ss} = 2 \end{aligned}$$

The steady-state output changed by 100 percent!

3.

$$\begin{aligned} S_{K_t} &= \lim_{s \rightarrow 0} \frac{\partial G}{\partial K_t} \cdot \frac{K_t}{G} \\ &= \lim_{s \rightarrow 0} \frac{1}{R_f B (T_m s + 1)} \cdot R_f B (T_m s + 1) \\ &= 1 \end{aligned}$$

Because the static sensitivity is 1, if the parameter K_t changes by some percentage x , the steady-state output ω_{ss} will also change by the same percentage.

4. The parameter K_t will typically vary from motor to motor, and even for a single motor, it will vary over time (e.g. as components wear down). Using open-loop control to control the speed of the motor shaft would require measuring K_t (and other parameters) very precisely every time the motor is operated!

Part 2:

1. Use the final value theorem again, but this time for the closed-loop transfer function:

$$\begin{aligned} \omega_{ss} &= \lim_{s \rightarrow 0} s R(s) \frac{KG(s)}{1 + KG(s)} \\ &= \lim_{s \rightarrow 0} s \cdot \frac{1}{s} \cdot \frac{KK_t}{R_f B T_m s + R_f B + KK_t} \\ &= \frac{KK_t}{R_f B + KK_t} \end{aligned}$$

2.

$$\begin{aligned} K_t = 1 &\Rightarrow \omega_{ss} = \frac{(1)(1)}{1 + (1)(1)} = \frac{1}{2} \\ K_t = 2 &\Rightarrow \omega_{ss} = \frac{(1)(2)}{1 + (1)(2)} = \frac{2}{3} \end{aligned}$$

This time, with closed-loop control, the steady-state output changes by only 33 percent.

3.

$$\begin{aligned} K_t = 1 &\Rightarrow \omega_{ss} = \frac{(10)(1)}{1 + (10)(1)} = \frac{10}{11} \\ K_t = 2 &\Rightarrow \omega_{ss} = \frac{(10)(2)}{1 + (10)(2)} = \frac{20}{21} \end{aligned}$$

With a higher value of gain, the steady-state output changes by only about 5 percent.

4. The closed-loop transfer function was calculated already:

$$T(s) = \frac{KG(s)}{1 + KG(s)} = \frac{KK_t}{R_f B T_m s + R_f B + KK_t}$$

So the static sensitivity to K_t is given by:

$$\begin{aligned}
S_{K_t} &= \lim_{s \rightarrow 0} \frac{\partial T}{\partial K_t} \cdot \frac{K_t}{T} \\
&= \lim_{s \rightarrow 0} \frac{(R_f B T_m s + R_f B + K K_t)(K) - (K K_t)(K)}{(R_f B T_m s + R_f B + K K_t)^2} \cdot \frac{R_f B T_m s + R_f B + K K_t}{K} \\
&= \frac{R_f B}{K(R_f B + K K_t)}
\end{aligned}$$

Substituting the given values:

$$\begin{aligned}
K &= 1 \Rightarrow S_{K_t} = \frac{1}{2} \\
K &= 10 \Rightarrow S_{K_t} = \frac{1}{110}
\end{aligned}$$

5. Closed-loop control is far superior because it reduces sensitivity to variations in motor parameters like K_t . Since these parameters cannot be controlled, and tend to vary by a certain amount, closed-loop control is critical if you need to accurately control the speed of the motor. We can also see that, for the DC motor, increasing the gain of the controller reduces the sensitivity function S_{K_t} and therefore reduces the impact of parameter variations.

Problem 5

1. This system is type 1, so the steady-state error for a unit ramp input is given by $1/K$, where K is the gain of the forward path transfer function. For a proportional controller $G_c(s) = K_c$, the forward path transfer function is:

$$G(s) = \frac{K_c}{s(s+2)} = \frac{K_c/2}{s} \cdot \frac{1}{\frac{s}{2} + 1}$$

So the gain of the system is $K_c/2$. To get a steady-state error of 0.1, therefore, we must have:

$$\frac{1}{K_c/2} = 0.1 \Rightarrow K_c = 20$$

2. To get a zero steady-state error for a unit ramp input, we must have a type 2 system. There is already one integrator in the plant, so the controller G_c should include another integrator.
3. First, find the transfer function between disturbance and output:

$$\frac{C(s)}{D(s)} = \frac{\frac{1}{s(s+2)}}{1 + \frac{G_c(s)}{s(s+2)}} = \frac{1}{s(s+2) + G_c(s)}$$

So the steady-state output in response to a unit step disturbance input is:

$$c_{ss} = \lim_{s \rightarrow 0} \frac{sD(s)}{s(s+2) + G_c(s)} = \lim_{s \rightarrow 0} \frac{s \cdot \frac{1}{s}}{s(s+2) + G_c(s)} = \lim_{s \rightarrow 0} \frac{1}{G_c(s)}$$

So if $G_c(s)$ includes an integrator ($1/s$ term), then the output will have zero steady-state response to a disturbance, which is what we want!