16.06 Lecture 16

Controllability

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Today's Topics

1. Simple examples of controllable and uncontrollable systems
2. Definition of controllability
3. Controllability conditions for scalar inputs

The basic concept of controllability is that we can use inputs $\underline{u}(t)$ to drive the state $\underline{x}(t)$ to any point in the state space. The primary issue is whether the structure of the system makes this possible or impossible. Let's look at some simple examples to motivate the discussion

Example of an Aircraft Flying a Straight Path

We model distance along the path as the state x, and its derivative (speed along the path) as x_2. Out input is acceleration along the path

If the input is zero the trajectory in state space is a straight line (speed along the path is constant)

Inputs have the effect of directly changing speed x_2, and indirectly changing its integral x_1. For example, if inputs are constant.

Thus the control can drive the state from any initial point $x_1(0), x_2(0)$ to any final point $x_1(T), x_2(T)$. Furthermore, this can be accomplished in any specified time T. System is controllable!

Second Example of an Aircraft Flying a Straight Path

Now assume there is a prevailing wind which is constant. Wind velocity is another state in the problem

We now have

Speed along the path is no longer x_2 but rather is the sum of x_2 and x_3.

Our state space is now three dimensional. However, the control can only affect x_1 and x_2 but not x_3. We have no control over wind speed.

We are confined to the plane defined by $x_3(t) = x_3(0)$

The lack of controllability does not allow control of both airspeed and groundspeed independently. We can control one or the other but not both. This is particularly important for aircraft on landing and takeoff, which are always done heading into the wind, so as to maximize airspeed at a given groundspeed.

With these examples in mind let's proceed to define controllability

Controllability Definition

The system

is controllable if and only if there is always a control function $\underline{u}(t)$ that will drive the state from any initial point $\underline{x}(0)$ to any final point $\underline{x}(T)$ in a specified time T.

Now we have the general solution for $\underline{x}(T)$

which looks like this

Since $\underline{x}(T)$ can be anywhere in the space, for controllability it must be possible for the convolution integral

to reach any point in the space. First consider

Case 1: u(t) is a scalar

If $u(t)$ is a scalar, then the B matrix has only one column, which we denote as

Also, we know that the state transition matrix is a matrix exponential so

and our convolution integral becomes

Now define the following scalar quantities

So

This is just a sum of vectors. If the integral on the left is to be able to reach any point in the space then the sequence of vectors

must span the n dimensional state space. To have controllability it must be possible to find at least n linearly independent vectors in this sequence. Lets look back at our examples

Aircraft With No Wind

Thus $\underline{b}$ and $A\underline{b}$ are two linearly independent vectors, so they span the two dimensional state space, and the system is controllable.

Now let's consider the other example

Aircraft With Prevailing Wind

we have only two linear independent vectors for a three dimensional space. The system is uncontrollable.