16.06 QUIZ 2
November 19, 2003
Each question is worth an equal number of points.

Problem 1

Given the following system-

$$A=\begin{bmatrix}-3 & 1\\ 0 & 0\end{bmatrix} \qquad B=\begin{bmatrix}1\\ 1\end{bmatrix} \qquad C=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \qquad D=[0]$$

1.a) Determine the system state transition matrix

1.b) If the initial condition for the system state vector is-

$$x=\begin{bmatrix}0\\ 1\end{bmatrix}$$

and the input is a unit step function, determine the output by using convolution integrals.

(a) $$sI-A = \begin{bmatrix} s+3 & -1 \\ 0 & s\end{bmatrix}$$

$$\Rightarrow (sI-A)^{-1} = \frac{1}{s(s+3)}\begin{bmatrix} s & 1 \\ 0 & s+3\end{bmatrix} = \begin{bmatrix} \frac{1}{s+3} & \frac{1}{s(s+3)} \\ 0 & \frac{1}{s}\end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+3} & \frac{1/3}{s} - \frac{1/3}{s+3} \\ 0 & \frac{1}{s}\end{bmatrix}$$

$$\Phi(t) = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right]$$

$$= \begin{bmatrix} e^{-3t} & \frac{1}{3} - \frac{1}{3}e^{-3t} \\ 0 & 1\end{bmatrix}$$

(b) $\underset{\sim}{x}(t) = \Phi(t)\underset{\sim}{x}(0) + \int_0^t \Phi(t-\tau) B u(\tau) d\tau$

$$= \begin{bmatrix} e^{-3t} & \frac{1}{3}-\frac{1}{3}e^{-3t} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-3(t-\tau)} & \frac{1}{3}-\frac{1}{3}e^{-3(t-\tau)} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}(1)\, d\tau$$

$$= \begin{bmatrix} \frac{1}{3}-\frac{1}{3}e^{-3t} \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} \frac{1}{3}+\frac{2}{3}e^{-3(t-\tau)} \\ 1 \end{bmatrix} d\tau$$

$$= \begin{bmatrix} \frac{1}{3}-\frac{1}{3}e^{-3t} \\ 1 \end{bmatrix} + \left.\begin{bmatrix} \frac{1}{3}\tau+\frac{2}{9}e^{-3(t-\tau)} \\ \tau \end{bmatrix}\right|_{\tau=0}^{\tau=t}$$

$$= \begin{bmatrix} \frac{1}{3}-\frac{1}{3}e^{-3t} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{3}t+\frac{2}{9}-\frac{2}{9}e^{-3t} \\ t \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{9}+\frac{1}{3}t-\frac{5}{9}e^{-3t} \\ 1+t \end{bmatrix}$$

$$\underset{\sim}{y}(t) = C\underset{\sim}{x}(t) + \cancel{D}^{\,0} u(t)$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{5}{9}+\frac{1}{3}t-\frac{5}{9}e^{-3t} \\ 1+t \end{bmatrix}$$

$$= \begin{bmatrix} \frac{14}{9}+\frac{4}{3}t-\frac{5}{9}e^{-3t} \\ 1+t \end{bmatrix}$$

Problem 2

You are to design the attitude control system for a boost vehicle with a large and massive rocket motor. The transfer function relating $\theta(t)$, the pitch angle of the boost vehicle, to $u(t)$, the input command to the rocket motor actuator, is as follows-

$$\frac{\Theta(s)}{U(s)} = \frac{10}{(s+10)} \cdot \frac{(s^2+1)}{(s^2-1)}$$

where

$\Theta(s)$= Laplace Transform of the vehicle pitch angle
$U(s)$= Laplace Transform of the input to the rocket motor actuator

2.a) Create a state space model of this system

2.b) Develop a full state feedback control system so that the poles of the closed loop system are at-

$s = -5$
$s = -1 + j$
$s = -1 - j$

(a) Break into poles and zeros:

$$U(s) \longrightarrow \boxed{\frac{10}{(s+10)(s^2-1)}} \xrightarrow{X(s)} \boxed{s^2+1} \longrightarrow Y(s)$$

From the first block: $\dddot{x} + 10\ddot{x} - \dot{x} - 10x = 10u$

Define states: $\begin{cases} x_1 = x \\ x_2 = \dot{x} \\ x_3 = \ddot{x} \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = 10x_1 + x_2 - 10x_3 + 10u \end{cases}$

$$\Rightarrow \boxed{\dot{\underset{\sim}{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & 1 & -10 \end{bmatrix} \underset{\sim}{x} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u}$$

From the 2nd block: $y = \ddot{x} + x = x_3 + x_1$

$$\Rightarrow \boxed{y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \underset{\sim}{x}}$$

(b) Desired characteristic equation is:

$$(s+5)(s+1-j)(s+1+j) = 0$$

$$\Rightarrow \quad s^3 + 7s^2 + 12s + 10 = 0 \qquad ①$$

With full state feedback, the input is $u = u_c - Kx$

$$\Rightarrow \quad \dot{\underset{\sim}{x}} = (A - BK)\underset{\sim}{x} + Bu_c$$

$$A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & 1 & -10 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}[K_1 \ K_2 \ K_3] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10-10K_1 & 1-10K_2 & -10-10K_3 \end{bmatrix}$$

To get the char. eqn., use the shortcut for a matrix of the form $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}$, the char. eqn. is $s^3 + a_2 s^2 + a_1 s + a_0 = 0$

$$\Rightarrow \quad s^3 + (10+10K_3)s^2 + (10K_2 - 1)s + (10K_1 - 10) = 0 \qquad ②$$

Equate ① and ② and solve for K_1, K_2, K_3:

$$\begin{cases} 10K_1 - 10 = 10 \\ 10K_2 - 1 = 12 \\ 10 + 10K_3 = 7 \end{cases} \Rightarrow \begin{cases} K_1 = 2 \\ K_2 = 1.3 \\ K_3 = -0.3 \end{cases}$$

So the controller is:

$$\boxed{u = u_c - 2x_1 - 1.3x_2 + 0.3x_3}$$

Problem 3

Consider the plant

$$G(s) = \frac{2}{s^2+1}$$

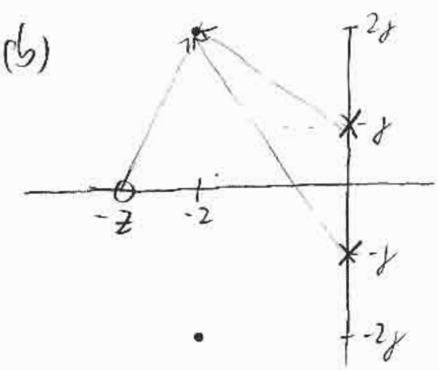
3.a) To achieve a second-order system that has a damping ratio of 0.707 and a time constant of 0.5s, where should the closed-loop poles be located?

3.b) Design a PD compensator which, when used in a unity feedback closed-loop system will meet the design specifications in 3.a). Be sure to state the final transfer function for your $G_c(s)$.

(a) $\zeta = \frac{1}{\sqrt{2}}$, $T = 0.5 = \frac{1}{\zeta \omega_n} \Rightarrow \omega_n = 2\sqrt{2}$

poles at $-\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2} \Rightarrow \boxed{-2 \pm 2j}$

(b)



PD compensator: $G_c(s) = K_c(s+z)$

• Apply angle condition to point $(-2+2j)$ to find z

$$\tan^{-1}\left(\frac{2}{z-2}\right) - \left(90° + \tan^{-1} 2\right) - \left(90° + \tan^{-1}\left(\tfrac{2}{3}\right)\right) = -180°$$

$$\Rightarrow \quad \frac{2}{z-2} = -8 \qquad \Rightarrow \quad \underline{z = \tfrac{7}{4}}$$

• Apply mag. condition to find K_c

$$K_{RL} = 2K_c = \frac{\left(\sqrt{1^2+2^2}\right)\left(\sqrt{3^2+2^2}\right)}{\sqrt{2^2 + (1/4)^2}} = 4 \quad \Rightarrow \quad \underline{K_c = 2}$$

$$\Rightarrow G_c(s) = 2(s + 7/4) = \boxed{2s + 3.5}$$

Problem 4

The plant

$$G(s) = \frac{(s^2+2s+2)}{s(s^2+s+0.5)}$$

has zeroes at $s=-1\pm j$ and poles at $s=0,\ s=-0.5\pm 0.5j$.

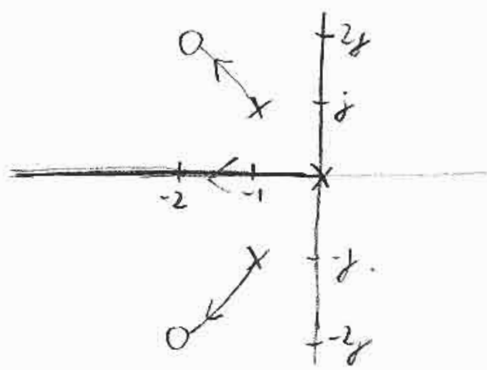
Consider the root locus plot for unity feedback.

4.a) Calculate the angle of departure from the upper complex pole for K>0

4.b) Calculate the angle of arrival at the upper complex zero for K>0

4.c) Sketch the root locus for K>0.

4.d) On a separate plot, sketch the root locus for K<0.



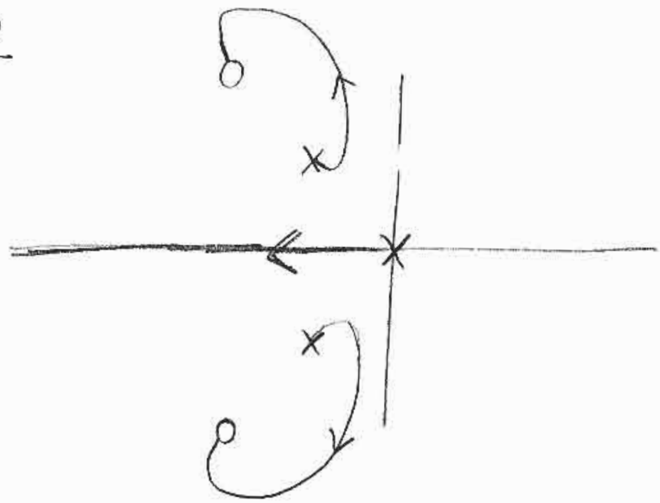
(a) Angle condition: $\left[-45^\circ + \tan^{-1}(3)\right] - \left[\gamma + 90^\circ + 135^\circ\right] = -180^\circ$

$$\Rightarrow \quad \boxed{\gamma = -18.4^\circ}$$

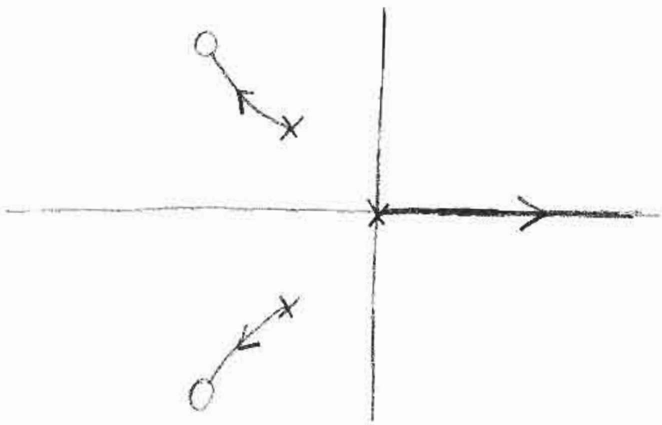
(b) Angle condition: $\left[\delta + 90^\circ\right] - \left[135^\circ + (90^\circ + \tan^{-1}(\tfrac{1}{3})) + 135^\circ\right] = -180^\circ$

$$\Rightarrow \quad \boxed{\delta = 108.4^\circ}$$

(c) $\underline{K > 0}$



(d) $\underline{K < 0}$



(angles of departure and arrival shifted by 180°)