16.060 Problem Set 9 Solutions

Fall 2003

Problem 1

1.) $y(t) = u(t-\tau)$

Using the definition of the Laplace transform:

$$Y(s) = \int_0^\infty e^{-st} u(t-\tau)\, dt$$

Use a change of variable: $T = t - \tau$

$$\Rightarrow Y(s) = \int_0^\infty e^{-s(T+\tau)} u(T)\, dT$$

$$= e^{-s\tau} \int_0^\infty e^{-sT} u(T)\, dT$$

$$= e^{-s\tau} U(s)$$

$$\Rightarrow \boxed{G(s) = e^{-s\tau}}$$

2.) $\tau = 1 \Rightarrow G(s) = e^{-s}$

For the polar plot: $s = j\omega$, $0 < \omega < \infty$

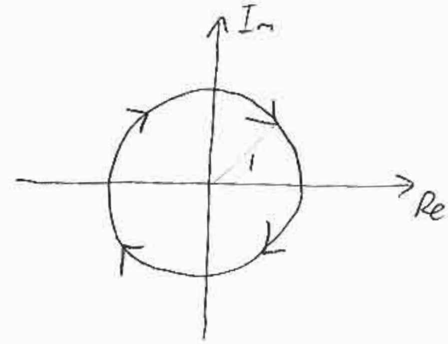
$$\Rightarrow G(s) = G(j\omega) = e^{-j\omega} = \cos\omega - j\sin\omega$$

$$\Rightarrow |G(j\omega)| = \sqrt{\cos^2\omega + \sin^2\omega} = 1$$

$$\angle G(j\omega) = -\omega$$

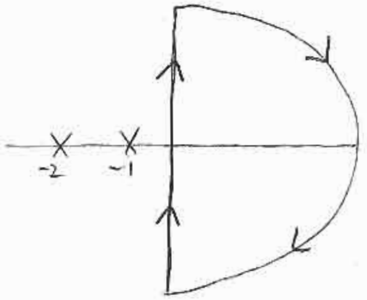
So the magnitude of G stays constant and equal to one, but the phase of G decreases from 0 to $-\infty$ as ω varies from 0 to ∞.

→ polar plot is a unit circle with infinite revolutions.

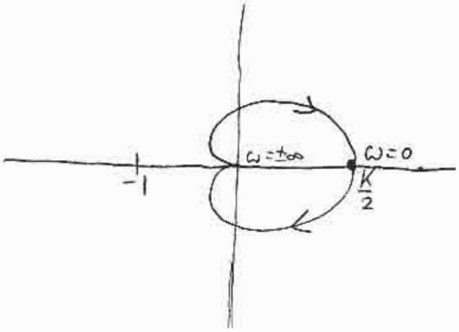


Problem 2

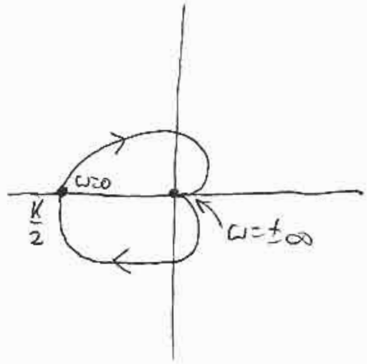
1.) $G(s) = \frac{K}{(s+1)(s+2)}$



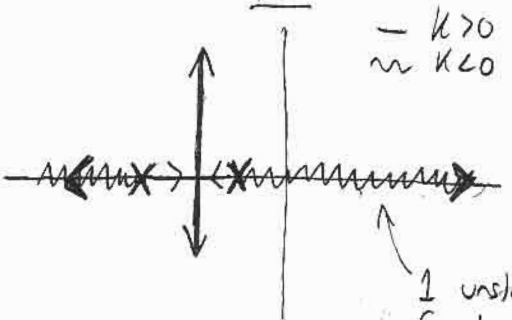
Nyquist $K>0$



Nyquist $K<0$



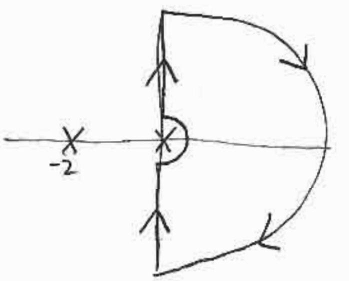
RL



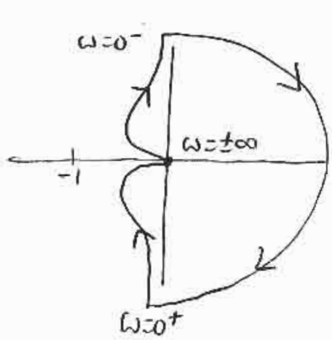
1 unstable pole
for large, negative K
stable otherwise

K	P	N	Z	Stable?
$K<-2$	0	1	1	No
$-2<K<0$	0	0	0	Yes
$K>0$	0	0	0	Yes

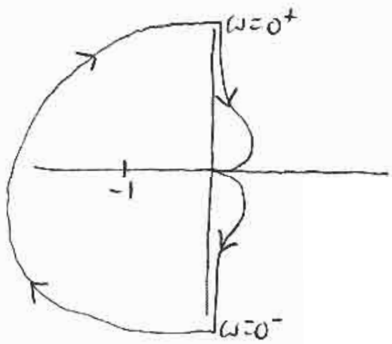
2.) $G(s) = \frac{K}{s(s+2)}$



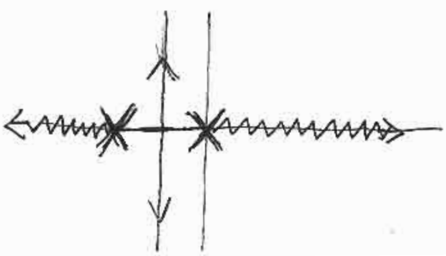
Nyquist $K>0$



Nyquist $K<0$

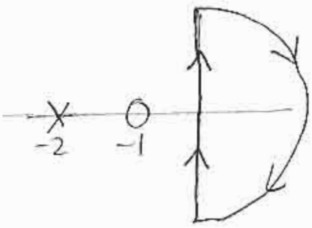


RL

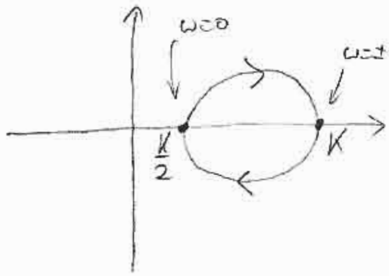


K	P	N	Z	Stable?
$K<0$	0	1	1	No
$K>0$	0	0	0	Yes

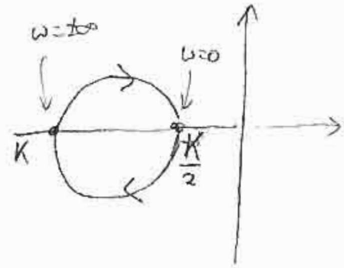
3.) $G(s) = \dfrac{K(s+1)}{s+2}$



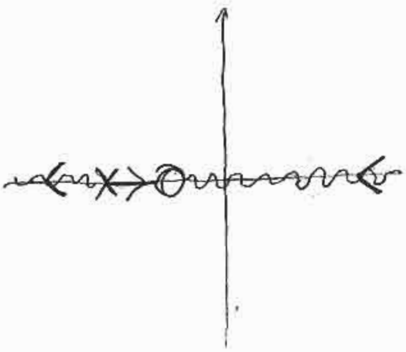
Nyquist $K>0$



Nyquist $K<0$

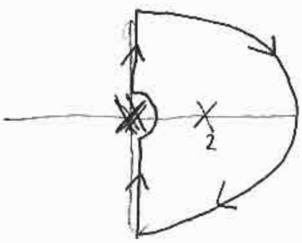


RL

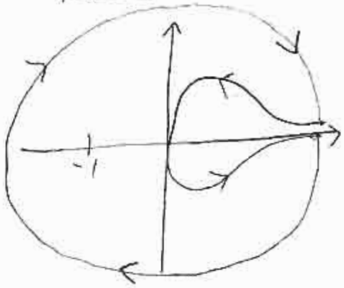


K	P	N	Z	Stable?
$K<-2$	0	0	0	Yes
$-2<K<-1$	0	1	1	No
$-1<K<0$	0	0	0	Yes
$K>0$	0	0	0	Yes

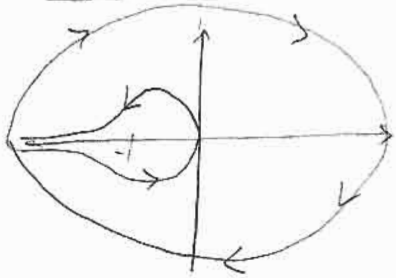
4.) $G(s) = \dfrac{K}{s^2(s-2)}$



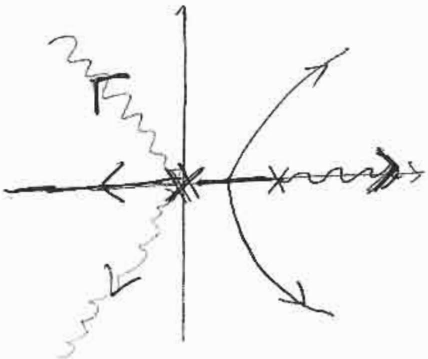
Nyquist $K>0$



Nyquist $K<0$



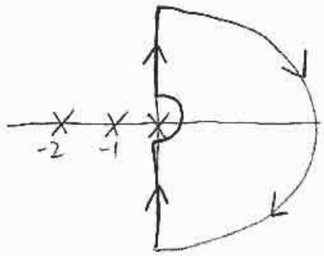
K	P	N	Z	Stable?
$K<0$	1	0	1	No
$K>0$	1	1	2	No



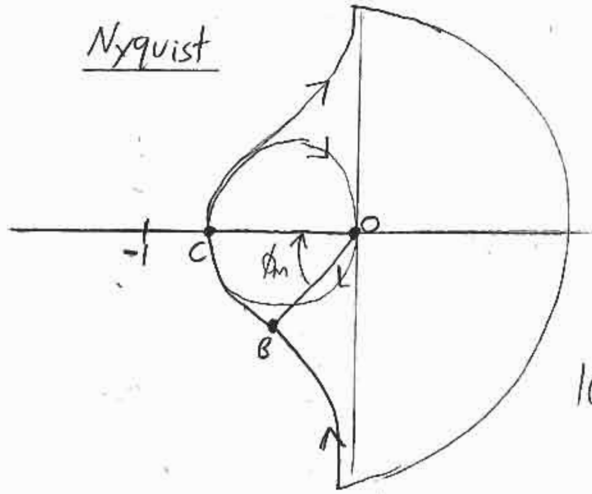
Problem 3

$$G(s) = \frac{3}{s(s+1)(s+2)}$$

1.)

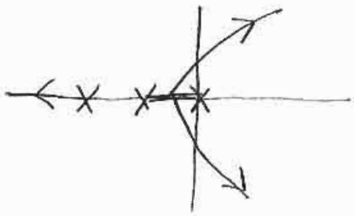


Nyquist



$|OB| = 1$

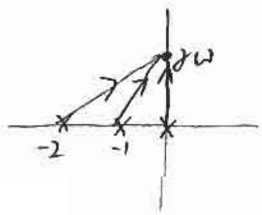
Root Locus



Gain margin $= \dfrac{1}{|OC|}$

Phase margin $= \measuredangle BOC$, where B is the point on the Nyquist plot for $\omega > 0$ such that $|OB| = 1$.

2.) To determine stability, we can calculate point C on the Nyquist plot. If it lies inside the -1 point, then $N=0$, $P=0$, $\Rightarrow Z=0$ and the CL system is stable.



Point C is where $\measuredangle G(j\omega) = -180°$

$\Rightarrow 90° + \tan^{-1}\left(\frac{\omega}{1}\right) + \tan^{-1}\left(\frac{\omega}{2}\right) = 180°$

$\Rightarrow \omega = \sqrt{2}$ (from Matlab)

$$\Rightarrow |G(j\omega)| = \frac{3}{(\sqrt{2})(\sqrt{1^2+2})(\sqrt{2^2+2})} = \frac{1}{2}$$

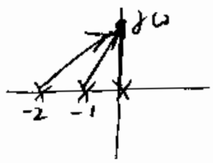
So the point C is at $\left(-\frac{1}{2}, 0\right)$, and therefore the CL system is $\boxed{\text{stable}}$

Gain margin:

$$GM = \frac{1}{|OC|} = \frac{1}{1/2} = \boxed{2}$$

Phase margin:

To find the phase margin, we must find point B on the Nyquist plot (where $|G(j\omega)| = 1$).



$$\Rightarrow \frac{3}{\omega\sqrt{1^2+\omega^2}\sqrt{2^2+\omega^2}} = 1$$

$$\Rightarrow \omega^2(1+\omega^2)(4+\omega^2) = 9$$

$$\Rightarrow \omega \approx 0.969 \quad \text{(from Matlab)}$$

$$\Rightarrow \angle G(j\omega) = -\left(90^\circ + \tan^{-1}(.969) + \tan^{-1}\left(\frac{.969}{2}\right)\right) = 159.9^\circ$$

$$\Rightarrow \phi_m = 180^\circ - 159.9^\circ = \boxed{20.1^\circ}$$

3.) the CL characteristic equation is:

$$s(s+1)(s+2) + K = 0$$

$$\Rightarrow s^3 + 3s^2 + 2s + K = 0$$

Substituting $s = j\omega$, $K = K_{crit}$:

$$(j\omega)^3 + 3(j\omega)^2 + 2j\omega + K_{crit} = 0$$

① Real part: $-3\omega^2 + K_{crit} = 0$ ② Im. part: $-\omega^3 + 2\omega = 0$

From ②: $\omega = \sqrt{2}$

Subs. into ①: $\boxed{K_{crit} = 6}$

Notice: the RL gain of G is 3, so $\boxed{K_{crit} = K_{RL} \cdot GM}$

ie. GM = the amount by which you can increase the gain of the system before the CL system goes unstable.

4.) Using the nichols( ) function in Matlab, I got the following table of values for $\angle G(j\omega)$ and $|G(j\omega)|$:

ω	$\angle G(j\omega)$	$\lvert G(j\omega)\rvert$ (dB)
.084	$-97°$	25
.134	$-101°$	21
.215	$-108°$	17
.346	$-119°$	12
.554	$-134°$	7
.889	$-156°$	1
1.43	$-180°$	-6
1.80	$-193°$	-10
2.29	$-205°$	-15
2.89	$-216°$	-20

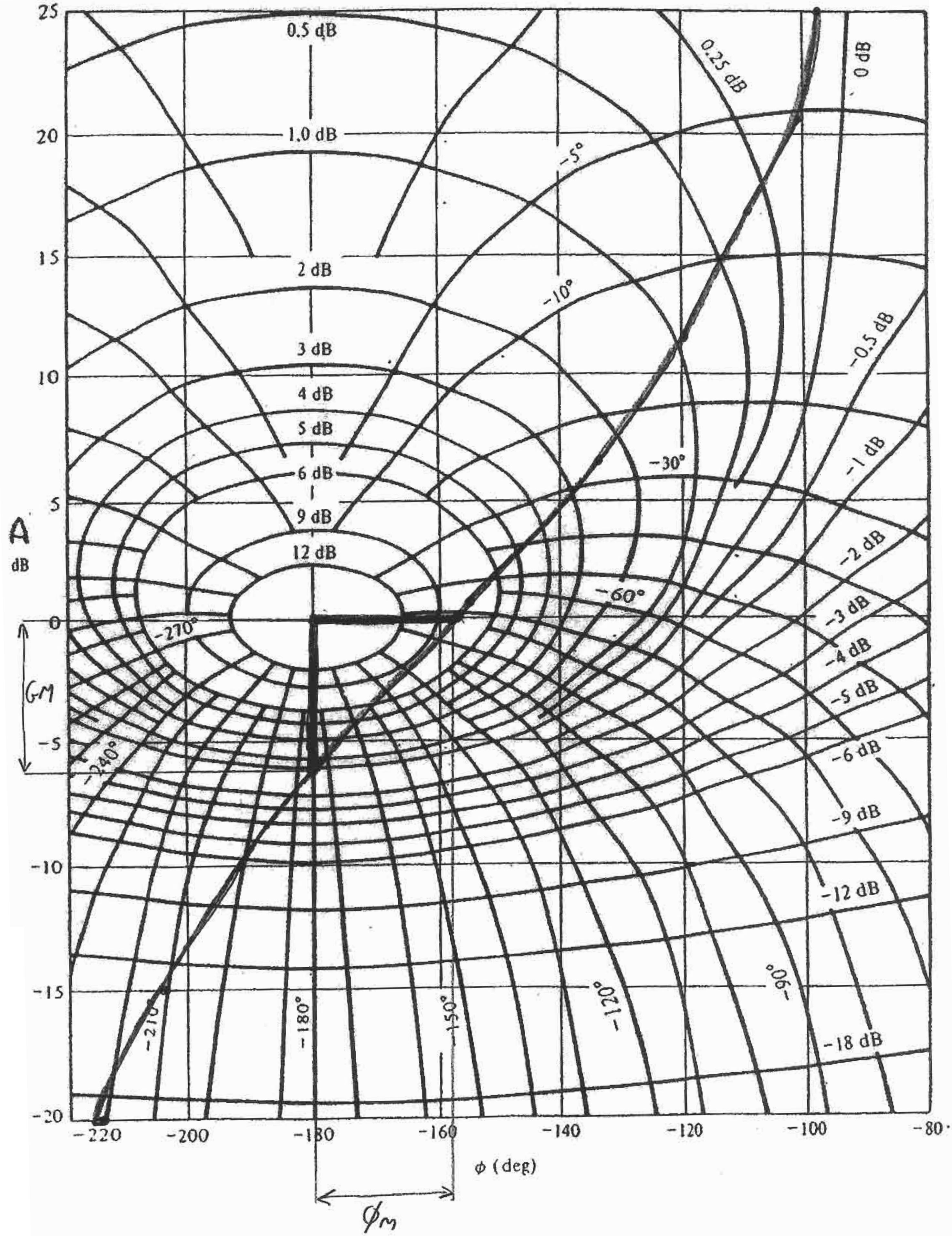
These data are sketched on the attached Nichols chart.

The GM is approximately 6 dB, or 2.

The phase margin is approximately 22°.

5.) Looking at the curve of $G(j\omega)$, and the contours for the magnitude of the CL system, it looks like the magnitude of the CL transfer function peaks at about 8 dB, or $\boxed{2.5}$. This point occurs where $|G(j\omega)| \approx -2$ dB, and $\angle G(j\omega) \approx -165°$, so the "peak frequency" is approximately $\boxed{\omega = 1.1}$, estimating from the table of values above.

6.) If GM and ϕ_m were higher, the curve of $G(j\omega)$ would stay further away from the -1 point, or ($-180°$, 0 dB) on the Nichols chart. Therefore, the peak amplitude of the CL system would be <u>lower</u>.



0.5 dB
1.0 dB
2 dB
3 dB
4 dB
5 dB
6 dB
9 dB
12 dB
0.25 dB
0 dB
−0.5 dB
−1 dB
−2 dB
−3 dB
−4 dB
−5 dB
−6 dB
−9 dB
−12 dB
−18 dB
−5°
−10°
−30°
−60°
−90°
−120°
−150°
−180°
−210°
−240°
−270°
A
dB
GM
ϕ (deg)
ϕM
25
20
15
10
5
0
−5
−10
−15
−20
−220
−200
−180
−160
−140
−120
−100
−80