MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Aeronautics and Astronautics
16.060: Principles of Automatic Control
Fall 2003

Quiz 2 Practice Problems
SOLUTIONS

State-Space

1. (a) The poles of the system are the eigenvalues of A:

$$\begin{aligned} |sI - A| &= \det \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} = s(s+3) + 2 = (s+1)(s+2) \\ \Rightarrow s &= -2, -1 \end{aligned}$$

The poles are in the left half-plane, so the system is stable.

(b) The transfer function matrix is given by the formula $\mathbf{G}(s) = C[sI - A]^{-1}B + D$.

$$\begin{aligned} (sI - A) &= \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \\ \Rightarrow |sI - A| &= s^2 + 3s + 2 \\ \Rightarrow [sI - A]^{-1} &= \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \end{aligned}$$

Hence

$$G(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s^2+3s+2} \\ \frac{s}{s^2+3s+2} \end{bmatrix}$$

(c) The state transition matrix is given by $\Phi(t) = \mathcal{L}^{-1}(sI - A)^{-1}$.
From Problem 1(a) above we know that

$$[sI - A]^{-1} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

Taking the inverse Laplace Transform:

$$\Phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

(d) To find the step response we use the following formula:

$$\mathbf{x} = \Phi(t)\mathbf{x}(\mathbf{0}) + \int_0^t \Phi(t - \tau) B \mathbf{u}(\tau) d\tau$$

Since $\mathbf{y} = \mathbf{x}$ in this case and $u(t) = 0$ for $t < 0$ and $u(t) = 1$ for $t > 0$, we have the following:

$$
\begin{aligned}
\mathbf{y} &= \Phi(t)\mathbf{x}(\mathbf{0}) + \int_0^t \Phi(t-\tau)B\mathbf{u}(\tau) \\
&= \begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&\quad + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)}-e^{-2(t-\tau)} & e^{-(t-\tau)}-e^{-2(t-\tau)} \\ -2e^{-(t-\tau)}+2e^{-2(t-\tau)} & -e^{-(t-\tau)}+2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1]\, d\tau \\
&= \begin{bmatrix} 3e^{-t}-2e^{-2t} \\ -3e^{-t}+4e^{-2t} \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-\tau)}-e^{-2(t-\tau)} \\ -e^{-(t-\tau)}+2e^{-2(t-\tau)} \end{bmatrix} d\tau \\
&= \begin{bmatrix} 3e^{-t}-2e^{-2t} \\ -3e^{-t}+4e^{-2t} \end{bmatrix} + \begin{bmatrix} e^{-t}\int_0^t e^{\tau}d\tau - e^{-2t}\int_0^t e^{2\tau}d\tau \\ -e^{-t}\int_0^t e^{\tau}d\tau + 2e^{-2t}\int_0^t e^{2\tau}d\tau \end{bmatrix} \\
&= \begin{bmatrix} 3e^{-t}-2e^{-2t} \\ -3e^{-t}+4e^{-2t} \end{bmatrix} + \begin{bmatrix} e^{-t}(e^{t}-1) - \frac{1}{2}e^{-2t}(e^{2t}-1) \\ -e^{-t}(e^{t}-1) + e^{-2t}(e^{2t}-1) \end{bmatrix} \\
\Rightarrow \mathbf{y}(t) &= \begin{bmatrix} \frac{1}{2}+2e^{-t}-\frac{3}{2}e^{-2t} \\ -2e^{-t}+3e^{-2t} \end{bmatrix}
\end{aligned}
$$

2. The controllability matrix is:

$$
\Gamma_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ 1 & c & b+c^2 \end{bmatrix}
$$

   The three columns of Γ_c are linearly independent for any a, b, and c, so the system is controllable for any a, b, c.

3. First find the state-space description of the system:

$$
G(s) = \frac{2}{s^2+4s+5} \iff \ddot{y} + 4\dot{y} + 5y = 2u
$$

   Define the states $x_1 = y$, $x_2 = \dot{y}$. So the state equations are:

$$
\begin{aligned}
\dot{x_1} &= x_2 \\
\dot{x_2} &= -5x_1 - 4x_2 + 2u \\
\Rightarrow \dot{\vec{x}} &= \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u
\end{aligned}
$$

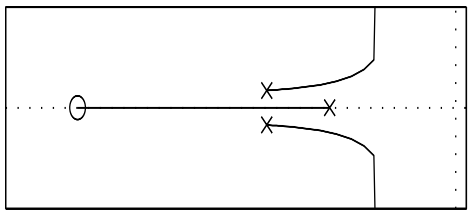
   For the closed-loop poles to be at $s = -3 \pm 2j$, the characteristic equation we want is $s^2 + 6s + 13 = 0$. The full-state feedback matrix is $k' = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$. The closed-loop dynamics are described by the matrix $A - Bk' = \begin{bmatrix} 0 & 1 \\ -5-2k_1 & -4-2k_2 \end{bmatrix}$. So the characteristic equation is also given by: $\det(sI - (A - Bk') = s^2 + (4+2k_2)s + (5+2k_1) = 0$. Equating coefficients, we get:

$$
\begin{aligned}
5 + 2k_1 = 13 &\Rightarrow k_1 = 4 \\
4 + 2k_2 = 6 &\Rightarrow k_2 = 1
\end{aligned}
$$

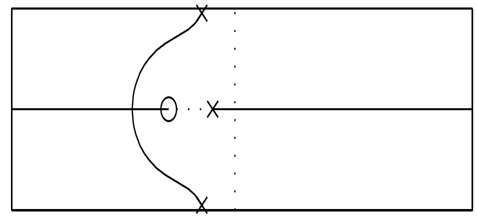
   So the feedback control is $u = 4x_1 + x_2$.

Root Locus

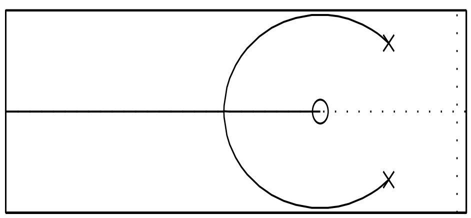
(1): K > 0



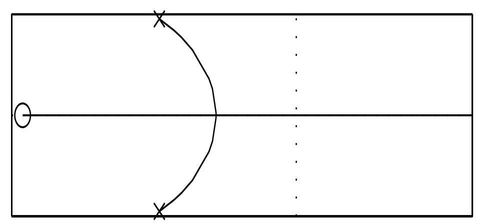
(1): K < 0



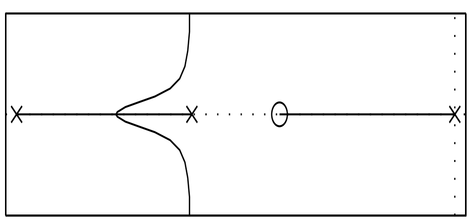
(2): K > 0



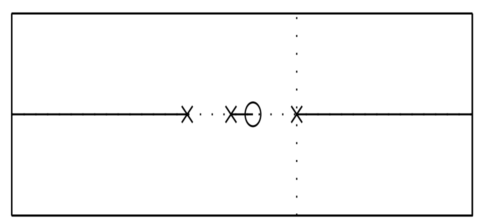
(2): K < 0



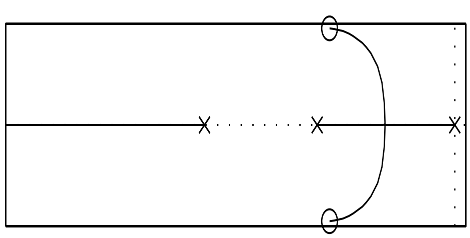
(3): K > 0



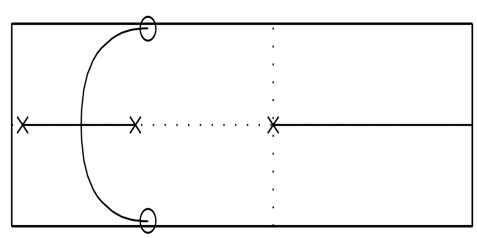
(3): K < 0



(4): K > 0



(4): K < 0



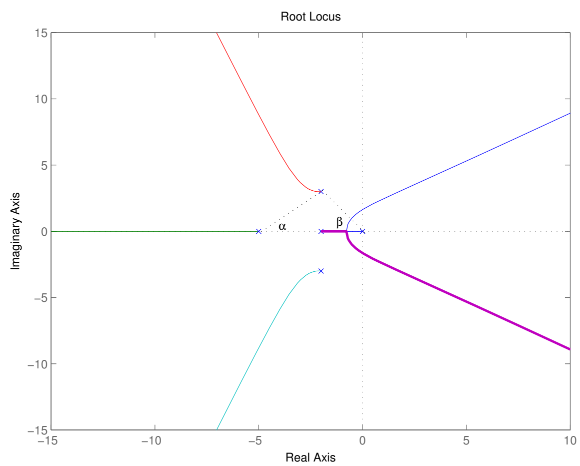
1.

2. (a) $G(s) = \frac{K}{s(s+2)(s+5)(s^2+4s+13)}$

- number of asymptotes: $n - m = 5$
- asymptotes: $\alpha = \frac{(2i+1)180°}{n-m} = \pm 36°, \pm 108°, 180°$
- $\rho_0 = \frac{0-2-5-2-2}{5} = -2.2$
- $-[\gamma + 90 + (180 - \beta) + \alpha + 90] = -180$ and using $\alpha = 45°$ and $\beta = 56°$ we get $\gamma = 191°$
- The closed-loop characteristic equation is $s(s+2)(s+5)(s^2+4s+13) + K = 0$ or $s^5 + 11s^4 + 51s^3 + 131s^2 + 130s + K = 0$. Substituting $s = j\omega$ and $K = K_{crit}$, we get $j\omega^5 + 11\omega^4 - 51j\omega^3 - 131\omega^2 + 130j\omega + K_{crit} = 0$ Breaking into real and imaginary parts:

$$\begin{aligned} 11\omega^4 - 131\omega^2 + K_{crit} &= 0 \\ \omega^5 - 51\omega^3 + 130\omega &= 0 \end{aligned}$$

  The second equation gives $\omega = 0, \pm 1.64, \pm 6.95$. Plugging back into the first equation gives $K_{crit} = 0, 273, -19337$. Since we are interested in the root locus for $K > 0$, the answer is $K_{crit} = 273$.
- The corresponding root locus is:



(b) $G(s) = \frac{K(s+5)}{s(s+2)(s^2+2s+5)}$

- number of asymptotes: $n - m = 3$
- asymptotes: $\alpha = \frac{(2i+1)180°}{n-m} = \pm 60°, 180°$
- $\rho_0 = \frac{0-2-1-1+5}{3} = \frac{1}{3}$
- $27 - \gamma - 90 - 117 - 63 = 180$ which gives $\gamma = -63°$
- The closed-loop characteristic equation is $s(s+2)(s^2+2s+5) + K(s+5) = 0$ or $s^4 + 4s^3 + 9s^2 + (K+10)s + 5K = 0$. Substituting $s = j\omega$ and $K = K_{crit}$, we get $\omega^4 - 4j\omega^3 - 9\omega^2 + (10 + K_{crit})j\omega + 5K_{crit} = 0$. Breaking into real and imaginary parts:

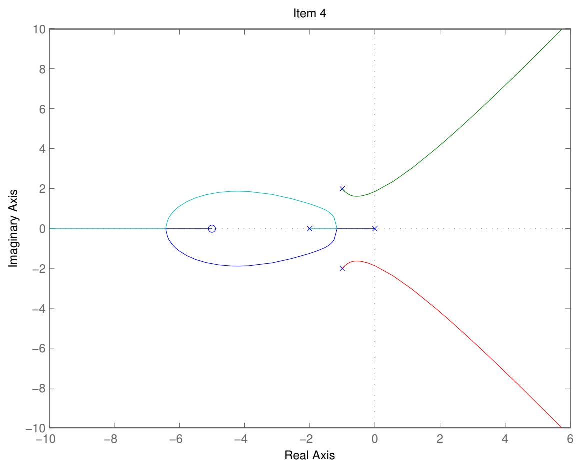
$$\begin{aligned} \omega^4 - 9\omega^2 + 5K_{crit} &= 0 \\ -4\omega^3 + (10 + K_{crit})\omega &= 0 \end{aligned}$$

  From the second equation, if we ignore the solution $\omega = 0$ (which corresponds to $K_{crit} = 0$), we get $\omega^2 = \frac{1}{4}(10 + K_{crit})$. Substituting back into the first equation gives:

$$\begin{aligned} \frac{1}{16}(10 + K_{crit})^2 - \frac{9}{4}(10 + K_{crit}) + 5K_{crit} &= 0 \\ \Rightarrow \frac{1}{16}K_{crit}^2 + 4K_{crit} - \frac{65}{4} &= 0 \\ \Rightarrow K_{crit} &= 3.83, -67.83 \end{aligned}$$

Since we're interested in the root locus for $K > 0$, the answer is $K_{crit} = 3.83$.

- The corresponding root locus is:



3. First, find the open-loop poles of the plant. The real pole has a time constant of 1/6 seconds, so that pole is at $s = -6$. The complex poles are at $s = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2} \approx -0.06 \pm 0.7j$.

   We want $\omega_n = 2.5$ rad/sec, $\zeta = 0.5 \Rightarrow$ closed loop poles at $-1.25 \pm 2.17j$. Please refer to Fig 1.

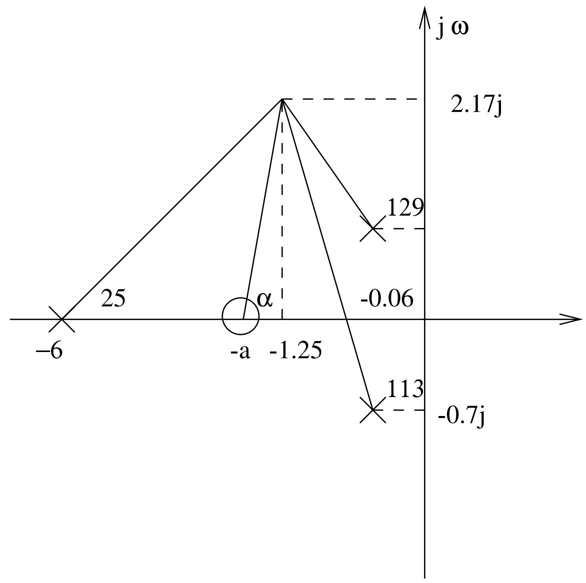


Figure 1: Pole-Zero Map

Apply the angle condition:

$$\alpha - 25^\circ - 129^\circ - 113^\circ = \pm 180^\circ$$

Which gives $\alpha = 87^\circ$ and using $\tan\alpha = \frac{2.17}{a-1.25}$ we get $a = 1.36$.

Apply the magnitude condition:

$$K_{rl} = \frac{(5.22)(1.87)(3.11)}{(2.17)} = 14$$

This is the total Root Locus gain of the system, $K_{rl} = K_{rl,plant} * K_d$. If K is the standard gain of the plant, ω_n is the undamped natural frequency of the open-loop complex poles, and T is the time constant of the open-loop real pole, then the root locus gain of the plant is given by $K\omega_n^2/T$ (you can check this result by writing out the transfer function of the plant. So $K_{rl,plant} = \frac{(0.4)(0.7)^2}{1/6} = 1.176$, and therefore $K_d = 14/1.176 = 11.9$.

The controller is then:

$$G_c(s) = 11.9(s + 1.36) = 11.9s + 16.2$$