# 16.06 Lecture 34

# Open and Closed Loop Behavior, Second Order System Paradigm

# November 26, 2003

Today's Topics:

1. Phase Margin vs Damping Ratio
2. Second Order System Model and Frequency Domain Criteria

When we were studying control system analysis in the time domain we used the second order system as a model to define various criteria such as percentage overshoot, rise time, settling time, etc. Although these criteria related specifically to second order systems we also found them useful for analyzing more complex systems. The basic assumption was that for most systems there is a dominant pole pair that serves to characterize the overall behavior of the larger, more complex system. We developed relationships for settling time, percentage overshoot and peak time in terms of the undamped natural frequency $\omega_n$ and damping ratio $\zeta$ of the second order system.

In analogous fashion we will relate the second order frequency response to undamped natural frequency and damping ratio and develop various criteria for evaluating closed loop system frequency responses.

First consider the following second order system with two real poles

and if we put unity feedback around the system we have the following root locus

The resulting closed loop system is

and we can immediately identify the open loop gain and the open loop pole location with the closed loop system undamped natural frequency and damping ratio

So we could have written the original open loop transfer function as

Now let's look at this feedback system in terms of its Nyquist diagram

The magnitude and phase of the open loop system are

where the angles $\alpha$ and $\beta$ are defined as shown in the diagram above. Now let's examine the Nyquist plot in the region near –1. The diagram is

As shown in the diagram, phase margin $\phi_m$ is the amount of phase shift necessary to make the Nyquist contour just cross the –1 point. It is determined by the arc from the –1 point to the point on the Nyquist diagram where the magnitude of the open loop transfer function is unity.

In order to determine phase margin we need to find the particular input frequency for which the magnitude of the open loop frequency response is unity. It is called the crossover frequency because typically, at that point, the magnitude "crosses over" from values larger than one to values smaller than one. The symbol $\omega_c$ is used to designate the crossover frequency. In the case at hand we can obtain an expression for $\omega_c$ by setting the magnitude of the open loop frequency response to one

and solve the resulting quadratic equation for its positive real root to yield

Hence the phase of the open loop frequency response, at the crossover frequency, is

Substituting into the phase margin equation shows that for a second order closed loop system there is a direct relationship between damping ratio and phase margin

This relationship is illustrated in the diagram on the following page

As can be seen in the diagram, the relationship is almost linear and is approximately

So, for example, if we want the closed loop system response to a step input to respond like a second order system, with a damping ratio of about 0.5, then a phase margin in the range of 50 degrees is necessary.

Let us now consider the closed loop frequency response. Typically it will look somewhat like this

Where we define

$M_p$=magnitude of the resonant peak

$\omega_p$=frequency at which the resonant peak occurs

$\omega_b$=bandwidth

Note also that magnitude .707 corresponds to –3 db

The resonant peak $M_p$ is the maximum magnitude of the frequency response of the closed loop system. The frequency at which that peak occurs is designated as $\omega_p$ and is often called the peak frequency. Bandwidth $\omega_b$ is defined as the range of frequencies over which the magnitude equals at least .707 times its value at zero frequency.

To obtain the frequency at which the peak occurs we write the frequency response of the closed loop second order system

Differentiate with respect to frequency and set the result equal to zero to obtain

Substituting back into the magnitude equation yields

These expressions are plotted in the following figure

The system bandwidth is determined by setting the magnitude equation to .707, squaring both sides and solving the resulting quadratic equation for the ratio of bandwidth to undamped natural frequency to yield

This equation, along with the crossover frequency equation, are plotted in the following diagram. In addition, the ratio of crossover frequency to bandwidth is also plotted in the diagram

A useful observation is that the ratio of bandwidth to crossover frequency is approximately constant up to damping ratios of about 0.8

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