MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Aeronautics and Astronautics
16.060: Principles of Automatic Control
Fall 2003

PROBLEM SET 6
Solutions

Problem 1

1.

$$\Gamma = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

There are two states, so we need two independent columns to show controllability. But the controllability matrix has only one independent column, so the system is uncontrollable.

2.

$$\Gamma = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

There are three states, so we need to find three independent columns to show controllability. The controllability matrix includes the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, which are independent and span the state-space $\Re^3$, so the system is controllable.

Problem 2

1. (a) The characteristic equation of the open-loop system is:

$$\begin{aligned} \det(sI - A) = 0 \quad &\Rightarrow \quad s^3 + 9s^2 + 23s + 15 = 0 \\ &\Rightarrow \quad (s+5)(s+3)(s+1) = 0 \end{aligned}$$

So the poles of the open-loop system are at $s = -1$, $s = -3$, and $s = -5$.

(b) We are given that the closed-loop poles are at $s = -3$ and $s = -4 \pm 4j$, so we want the characteristic equation for the closed-loop system to be:

$$\begin{aligned} (s+3)(s^2 + 8s + 32) &= 0 \\ \Rightarrow s^3 + 11s^2 + 56s + 96 &= 0 \end{aligned}$$

In terms of the matrices A, B, and K, the characteristic equation is given by $\det(sI-(A-BK) = 0$. We have:

$$A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 & -23 & -9 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 - 2k_1 & -23 - 2k_2 & -9 - 2k_3 \end{bmatrix}$$

So the characteristic equation in terms of the feedback gains (k_1, k_2, k_3) is:

$$s^3 + (9 + 2k_3)s^2 + (23 + 2 * k_2)s + (15 + 2 * k_1) = 0$$

Equating the coefficients of the two versions of the characteristic equation, we get:

$$\begin{aligned} 15 + 2k_1 = 96 &\quad\Rightarrow\quad k_1 = 40.5 \\ 23 + 2k_2 = 56 &\quad\Rightarrow\quad k_2 = 16.5 \\ 9 + 2k_3 = 11 &\quad\Rightarrow\quad k_3 = 1 \end{aligned}$$

So the control input we should use is:

$$u = u_c - K\vec{x} = u_c - 40.5x_1 - 16.5x_2 - x_3$$

2. (a) Substituting the values for the various parameters gives:

$$\ddot{\theta} = 4\theta + 4\delta$$

Using the states $x_1 = \theta$ and $x_2 = \dot{\theta}$, input $u = \delta$, and output $y = \theta$, we get the state-space model:

$$\begin{aligned} \dot{\vec{x}} &= \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \vec{x} \end{aligned}$$

(b) The open-loop poles of the system are the solutions to the characteristic equation of A, which is $s^2 - 4 = 0$. So the open-loop poles are at $s = 2$ and $s = -2$. There is a pole in the right half-plane, so the system is unstable. Therefore, the rocket would probably crash if it was operated without some kind of closed-loop control.

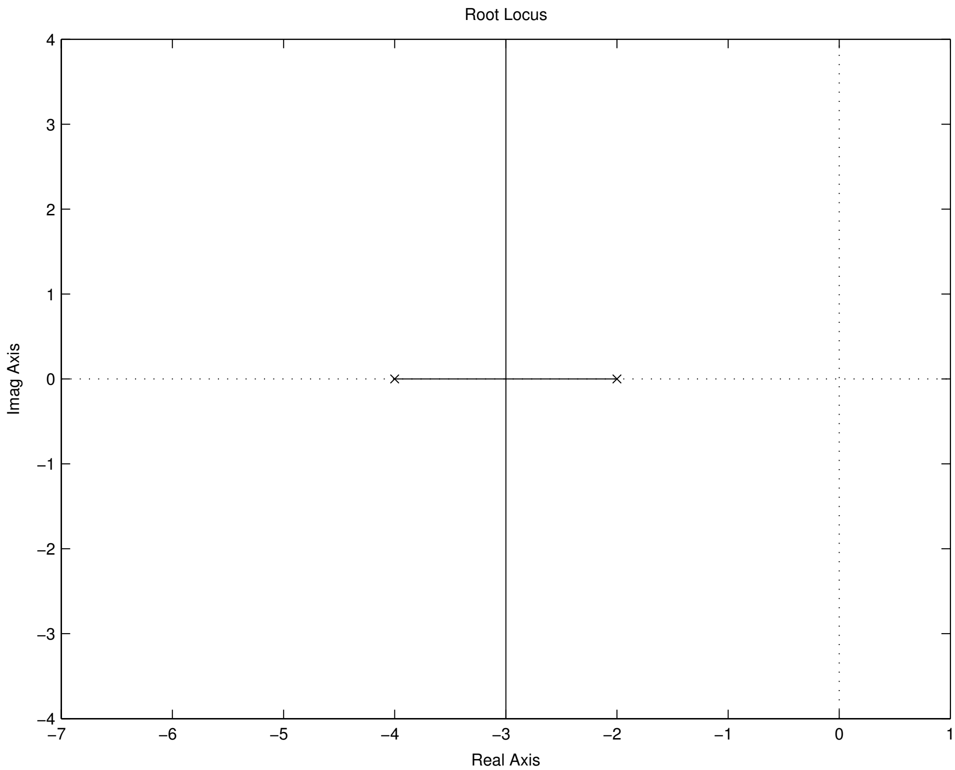
(c) The characteristic equation we want is $s^2 + 4s + 8 = 0$. For the feedback matrix $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, the closed-loop dynamics are described by the matrix $A - BK = \begin{bmatrix} 0 & 1 \\ 4 - 4k_1 & -4k_2 \end{bmatrix}$. So the characteristic equation is also given by: $s^2 + 4k_2 s + (4k_1 - 4) = 0$. Equating coefficients, we get:

$$\begin{aligned} 4k_2 = 4 &\quad\Rightarrow\quad k_2 = 1 \\ 4k_1 - 4 = 8 &\quad\Rightarrow\quad k_1 = 3 \end{aligned}$$

So the control input we should use is $u = 3x_1 + 4x_2$ or $u = 3\theta + 4\dot{\theta}$.

(d) In theory, we can calculate a feedback matrix K so that the closed-loop poles are in any location we like. But in practice, we run into physical limits. As we try to move the closed-loop poles further and further away from their open-loop positions, we need bigger and bigger values for the elements of K, and therefore the inputs to our physical plant also increase. Bigger inputs require more power (which is usually limited), and also may force the plant out of its linear operating regime. In this case, there is a physical limit to the angle δ at which we can deflect the rocket thrust (i.e. probably not more than 90 degrees) and also, for large values of δ, the original linearized dynamic model for the system is not valid so we can't reasonably expect our calculations to be valid either.

Problem 3



1.

2. The closed-loop transfer function is:

$$T(s) = \frac{K_c G}{1 + K_c G} = \frac{8K_c}{s^2 + 6s + (8 + 8K_c)}$$

   For the closed-loop system to be critically damped, the roots of $s^2 + 6s + (8 + 8K_c) = 0$ must be equal. Using the quadratic formula, we must have:

$$6^2 = 4(1)(8 + 8K_c) \Rightarrow K_c = \frac{1}{8}$$

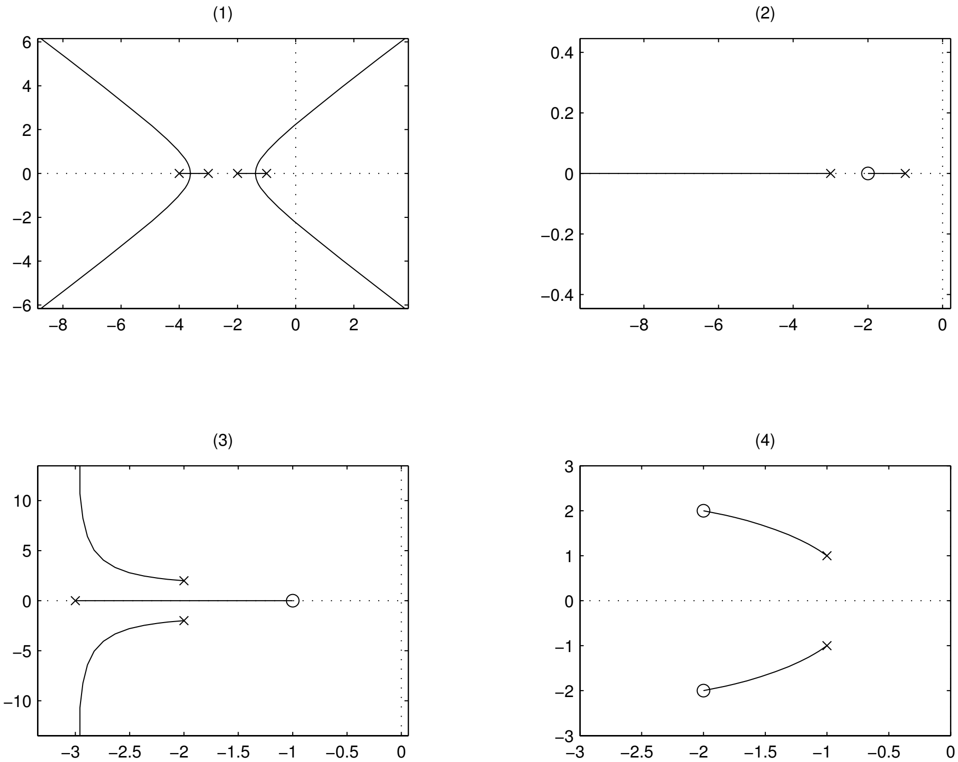
   Check: for $K_c = \frac{1}{8}$, the characteristic equation is $s^2 + 6s + 9 = 0$ which has repeated roots at $s = 3$.

3. A 2% settling time of $\frac{4}{3}$ corresponds to a time constant of $\frac{1}{3}$, so for complex conjugate poles, we must have $\zeta\omega_n = 3$. We are given $\zeta = \frac{1}{\sqrt{2}}$, so $\omega_n = 3\sqrt{2}$. So the desired characteristic equation is $s^2 + 2\zeta\omega_n s + {\omega_n}^2 = 0$, or $s^2 + 6s + 18 = 0$.

   Comparing the denominator of the closed-loop transfer function, we must have $8 + 8K_c = 18$ and so $K_c = 1.25$.

4. When $K_c = -1$, one of the poles of $T(s)$ is exactly at $s = 0$. For $K_c < -1$, this pole will be in the right half-plane, and the closed-loop system will be unstable.

Problem 4



To find the angle of departure in (4), apply the angle condition to a point very close to the pole at $s = -1 + j$. Let γ be the angle from that pole to the test point. The angle from the other pole to the test point is then 90°, the angle from the upper zero is -45°, and the angle from the lower zero is $\tan^{-1}(3) = 71.6^\circ$. So the angle condition specifies that:

$$-45^\circ + 71.6^\circ - \gamma - 90^\circ = -180^\circ \Rightarrow \gamma = 116.6^\circ$$

So the angle of departure of the locus for the upper pole is 116.6°. By symmetry, the angle of departure for the bottom pole must be -116.6°.