MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Aeronautics and Astronautics
16.060: Principles of Automatic Control
Fall 2003

PROBLEM SET 8
Solutions

Problem 1

The transfer function $G(s)$ can be written as:

$$G(s) = \frac{K_{rl}(s+z_1)(s+z_2)\cdots}{(s+p_1)(s+p_2)\cdots}$$

where K_{rl} is the root locus gain of $G(s)$, $-z_1, -z_2, \ldots$ are the zeros of $G(s)$, and $-p_1, -p_2, \ldots$ are the poles of $G(s)$. (Assume the system is stable, so all the poles lie in the left half-plane.)

The input to the system in the Laplace domain is:

$$R(s) = \frac{A\omega}{(s+j\omega)(s-j\omega)}$$

So the output is:

$$C(s) = G(s)R(s) = \frac{AK_{rl}(s+z_1)(s+z_2)\cdots}{(s+j\omega)(s-j\omega)(s+p_1)(s+p_2)\cdots}$$

Expanding into partial fractions:

$$\begin{aligned} C(s) &= \frac{K_1}{s+j\omega} + \frac{K_2}{s-j\omega} + \frac{K_3}{s+p_1} + \frac{K_4}{s+p_2} + \cdots \\ \Rightarrow c(t) &= K_1 e^{-j\omega t} + K_2 e^{j\omega t} + K_3 e^{-p_1 t} + K_4 e^{-p_2 t} + \cdots \end{aligned}$$

Since the system is stable, the terms $e^{-p_1 t}, e^{-p_2 t}, \ldots$ all decay to zero as $t \to \infty$. So the steady-state output is:

$$c_{ss} = \lim_{t\to\infty} c(t) = K_1 e^{-j\omega t} + K_2 e^{j\omega t}$$

Now we need to find the residuals K_1 and K_2. Using the coverup method, we get:

$$\begin{aligned} K_1 &= C(s)(s+j\omega)|_{s=-j\omega} = \left[\frac{A\omega(s+j\omega)}{(s+j\omega)(s-j\omega)} \cdot G(s)\right]_{s=-j\omega} = \frac{AG(-j\omega)}{-2j} \\ K_2 &= C(s)(s-j\omega)|_{s=j\omega} = \left[\frac{A\omega(s-j\omega)}{(s+j\omega)(s-j\omega)} \cdot G(s)\right]_{s=j\omega} = \frac{AG(j\omega)}{2j} \end{aligned}$$

Substituting back into the equation for c_{ss}, we get the steady-state output is:

$$c_{ss} = \frac{AG(-j\omega)e^{-j\omega t}}{-2j} + \frac{AG(j\omega)e^{j\omega t}}{2j}$$

Now $G(j\omega)$ is a complex number, that can be rewritten in terms of a magnitude and a direction:

$$G(j\omega) = M(\omega)e^{j\phi(\omega)}$$

where $M(\omega) = |G(j\omega)|$ and $\phi(\omega) = \angle G(j\omega)$. Since all complex zeros and poles of $G(s)$ come in conjugate pairs (as they do for any real-life transfer function):

$$G(-j\omega) = M(\omega)e^{-j\phi(\omega)}$$

Now substitute again into the equation for c_{ss}:

$$c_{ss} = \frac{A}{2j}(-M(\omega)e^{-j(\omega t+\phi(\omega))} + M(\omega)e^{j(\omega t+\phi(\omega))})$$

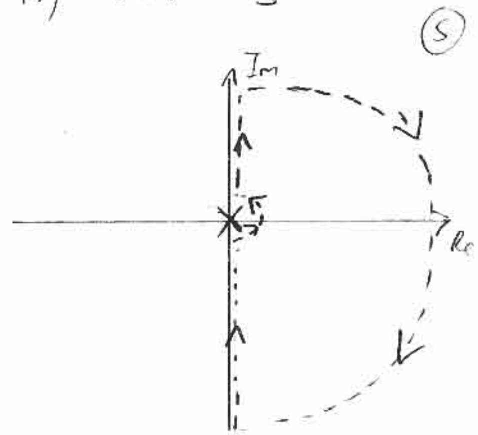
But $e^{j\theta} = \cos\theta + j\sin\theta$ and $e^{-j\theta} = \cos\theta - j\sin\theta$, so we get:

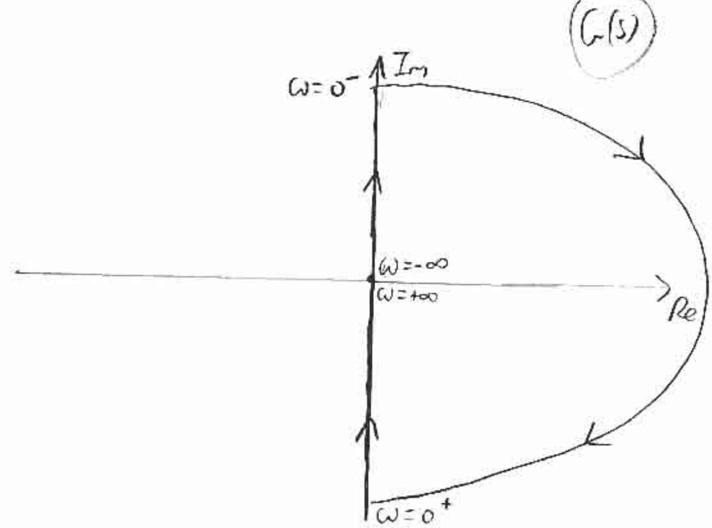
$$\begin{aligned} c_{ss} &= \frac{A}{2j}(2jM(\omega)\sin(\omega t + \phi(\omega))) \\ &= AM(\omega)\sin(\omega t + \phi(\omega)) \end{aligned}$$

$$A\sin(\omega t) \longrightarrow \boxed{G(s)} \longrightarrow AM(\omega)\sin(\omega t + \phi(\omega))$$

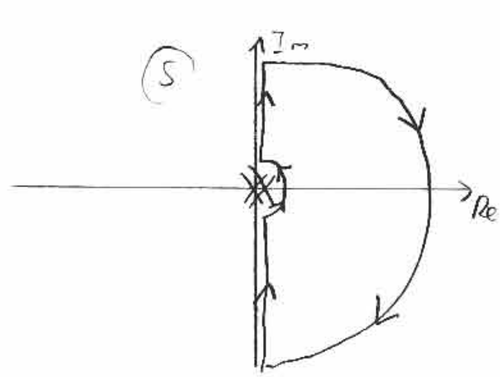
Problem 2

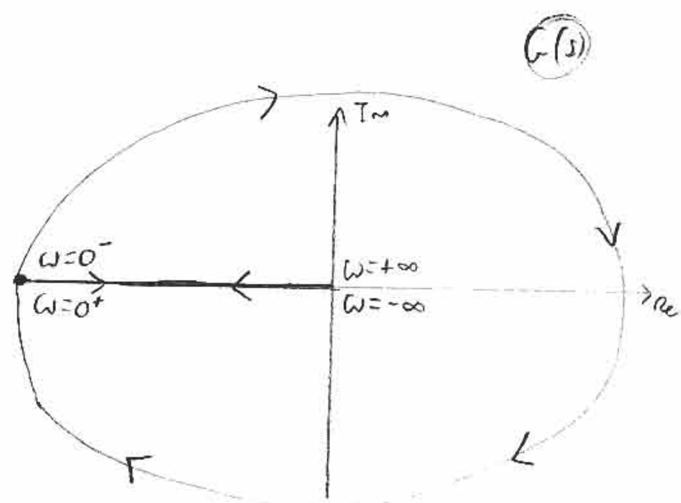
1.) $G(s) = \frac{K}{s}$



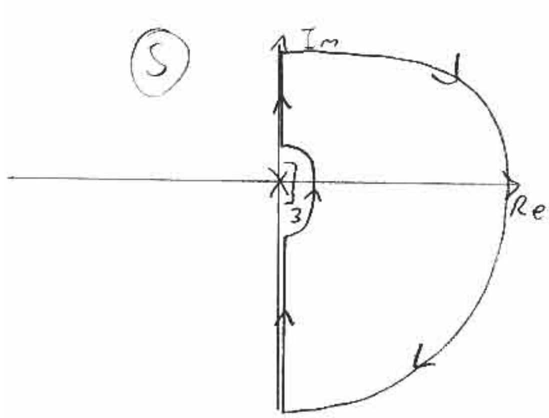


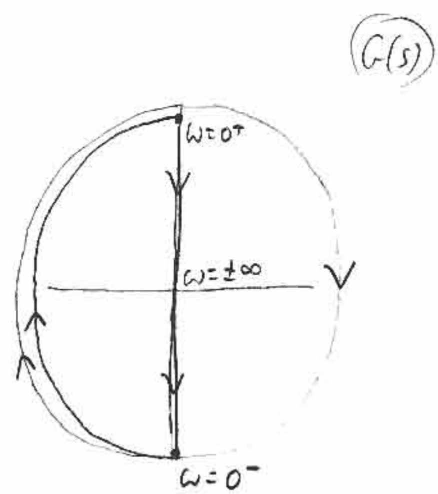
2.) $G(s) = \frac{K}{s^2}$



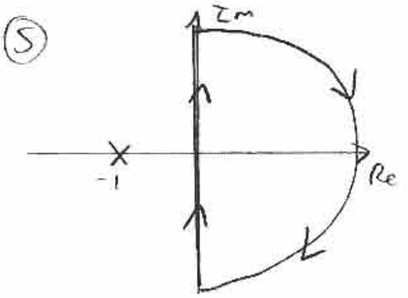


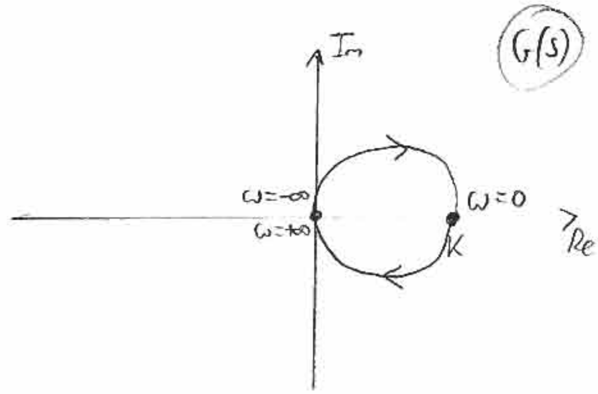
3.) $G(s) = \frac{K}{s^3}$



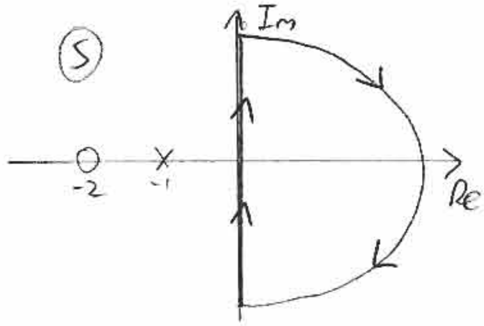


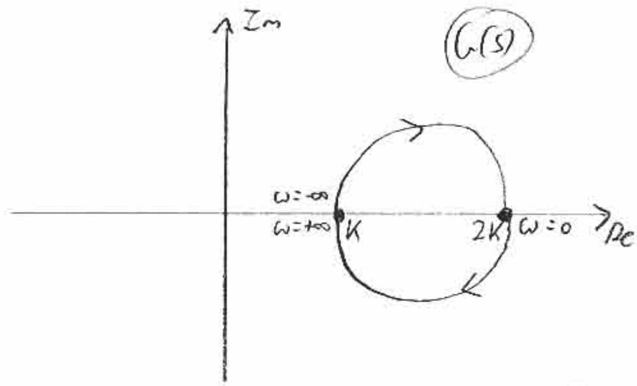
4.) $G(s) = \frac{K}{s+1}$



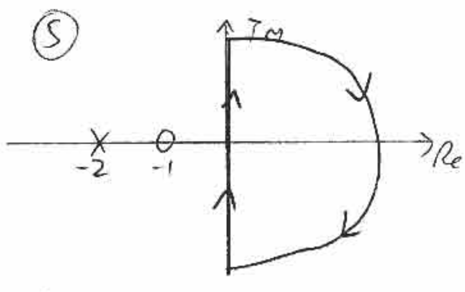


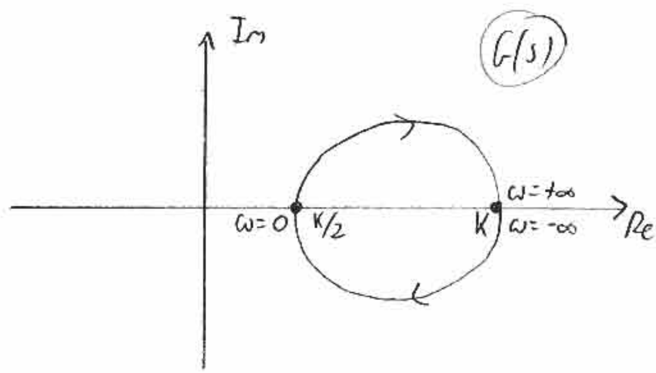
5.) $G(s) = \frac{K(s+2)}{s+1}$



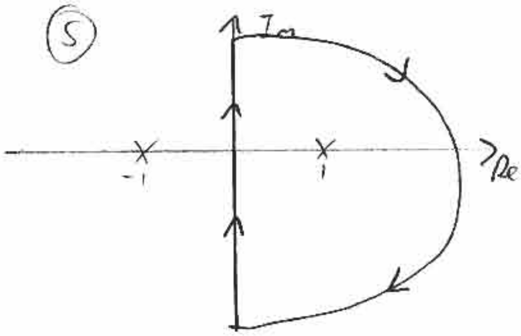


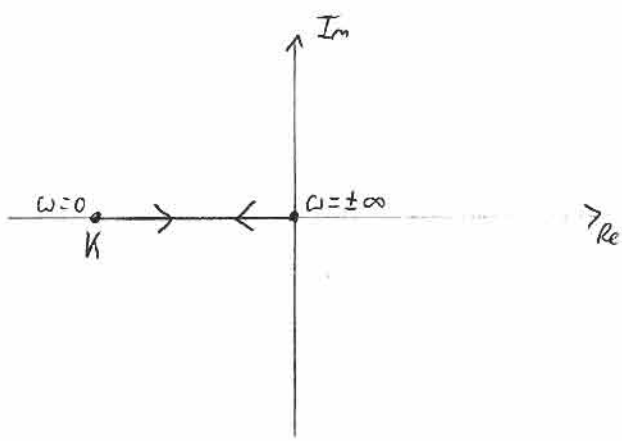
6.) $G(s) = \frac{K(s+1)}{s+2}$





7.) $G(s) = \frac{K}{(s+1)(s-1)}$

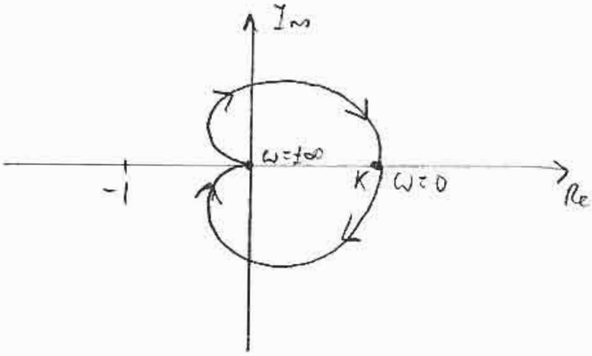




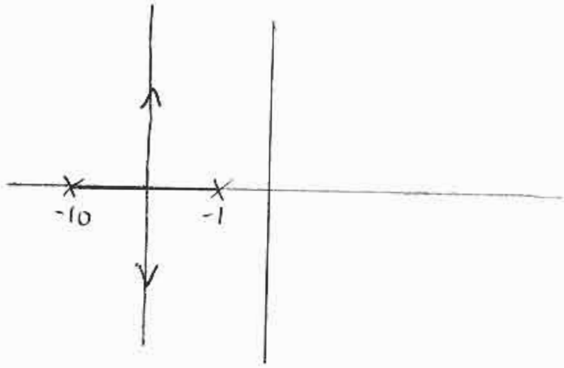
Problem 3

1)

(a) $G(s) = \dfrac{K}{(s+1)(\frac{s}{10}+1)} = \dfrac{10K}{(s+1)(s+10)}$



Nyquist



Root Locus

P = no. of OL poles in RHP = 0

N = no. of clockwise encirclements of -1 = 0 for all $K>0$

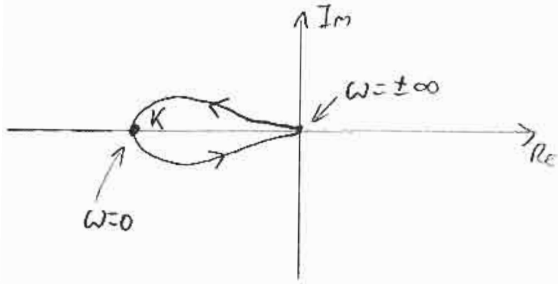
Z = no. of CL poles in RHP = $N+P$

$\Rightarrow \quad Z = N+P = 0$

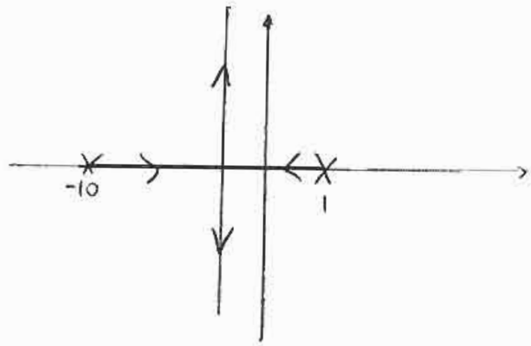
$\Rightarrow$ the CL system is stable for all $\boxed{K>0}$

→ Root Locus plot agrees.

(b) $G(s) = \dfrac{K}{(s-1)\left(\frac{s}{10}+1\right)} = \dfrac{10K}{(s-1)(s+10)}$



Nyquist



Root Locus

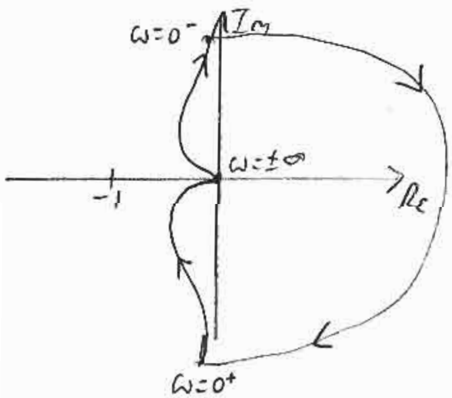
$P = 1$ (pole in RHP @ $s = 1$)

$N = \begin{cases} 0, & 0 < K \leq 1 \\ -1, & K > 1 \end{cases}$

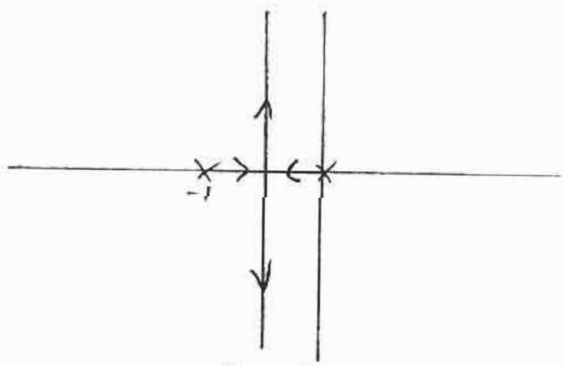
$\Rightarrow Z = N + P = \begin{cases} 1, & 0 < K < 1 \\ 0, & K \geq 1 \end{cases}$

$\Rightarrow$ the CL system is stable for $\boxed{K \geq 1}$
but one unstable pole when $0 < K < 1$

(c) $G(s) = \dfrac{K}{s(s+1)}$



Nyquist



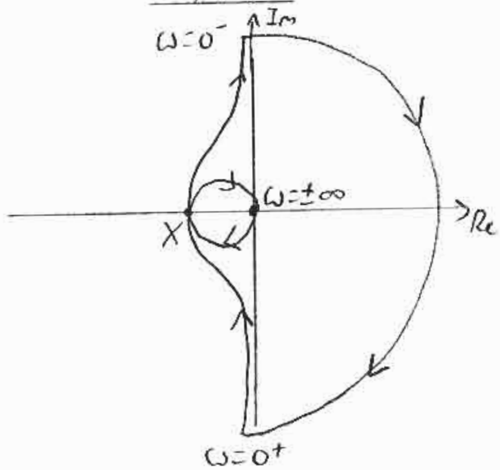
Root Locus

$P = 0$, $N = 0$ for all $K > 0$ $\Rightarrow Z = N + P = 0$ for all $K > 0$

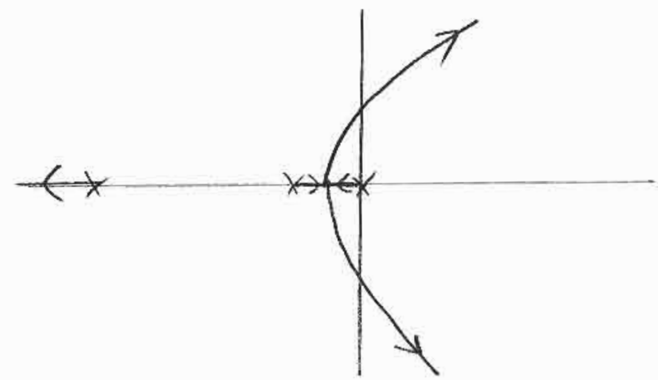
$\Rightarrow$ the CL system is stable for $\boxed{K > 0}$

(d) $G(s) = \dfrac{K}{s(s+1)(\frac{s}{10}+1)} = \dfrac{10K}{s(s+1)(s+10)}$

Nyquist

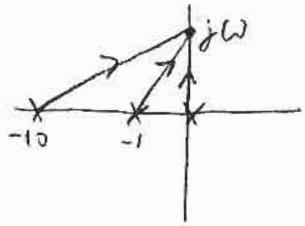


Root Locus



$P = 0$, $N = 0$ or 2, depending on the value of K.

We need to calculate the point X on the Nyquist diagram to determine the number of encirclements of the -1 point.
(Equivalently: calculate K_{crit} on the RL plot)



At point X: $\angle G(j\omega) = -180°$

$\Rightarrow \; -\left(90° + \tan^{-1}(\omega) + \tan^{-1}\left(\frac{\omega}{10}\right)\right) = -180°$

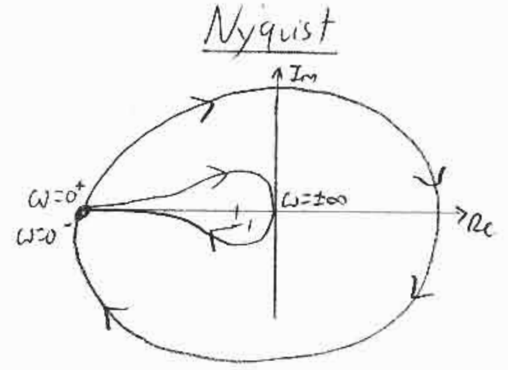
$\Rightarrow \; \omega = \sqrt{10}$ (from Matlab)

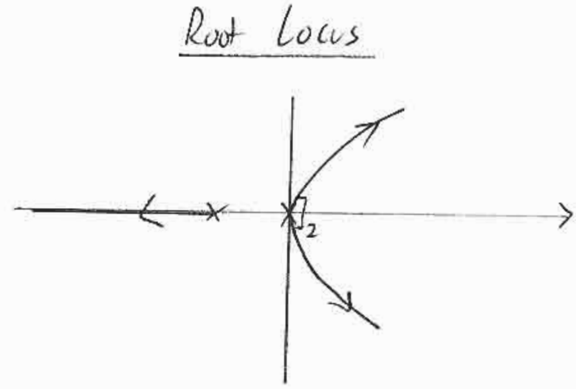
$\Rightarrow \; |G(j\omega)| = \dfrac{10K}{(\sqrt{10})(\sqrt{10+1^2})(\sqrt{10+10^2})} = \dfrac{K}{11}$ "X"

So if $0 < K < 11$, the point X lies to the <u>right</u> of the -1 point, and there are <u>zero</u> encirclements of -1. ($N=0 \Rightarrow Z=0$)
If $K > 11$, the point X lies to the <u>left</u> of the -1 point, and there are <u>two</u> encirclements of -1. ($N=2 \Rightarrow Z=2$)

$\Rightarrow$ the CL system is stable for $\boxed{0 < K \le 11}$

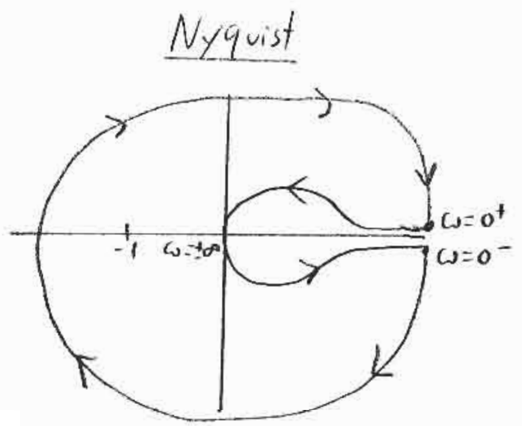
(e) $G(s) = \dfrac{K}{s^2(s+1)}$

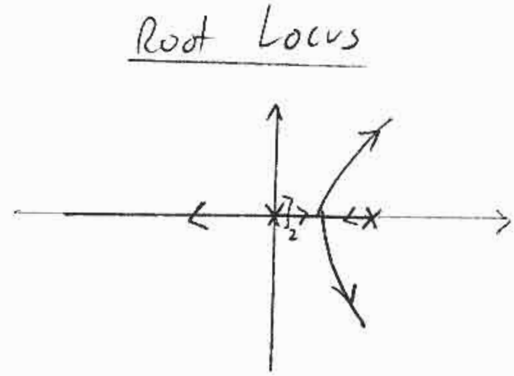




$N = 2$ for all $K > 0$
$P = 0$
$\Rightarrow Z = 2$ for all $K > 0$
$\Rightarrow$ the CL system is <u>unstable for all $K > 0$</u>

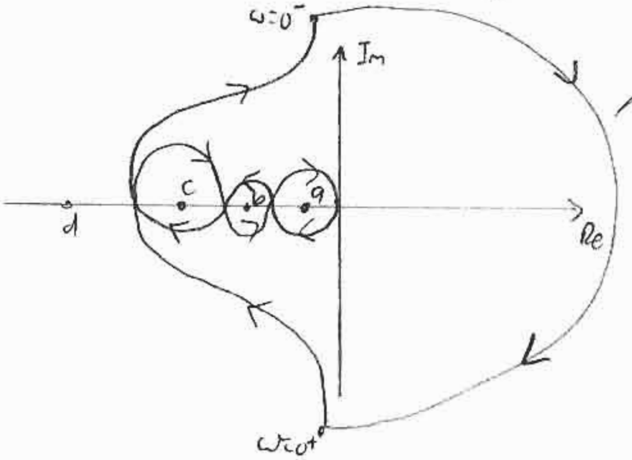
(f) $G(s) = \dfrac{K}{s^2(s-1)}$





$N = 1$ for all $K > 0$
$P = 1$
$\Rightarrow Z = 2$ for all $K > 0$
$\Rightarrow$ the CL system is <u>unstable for all $K > 0$</u>

2.) Complete Nyquist diagram:



One pole at origin

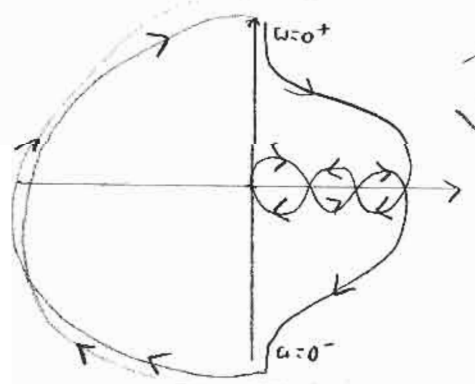
so $\measuredangle G(j\omega)$ changes by $-180°$ as ω goes from 0^- to 0^+

· We are given $P=0$ since all poles are in the LHP.

Position of −1 point	P	N	Z	Stable?
a	0	2	2	No
b	0	0	0	Yes
c	0	2	2	No
d	0	0	0	Yes

For very small gain, the −1 point is at (d), so the system is stable.
For very large gain, the −1 point is at (a), so the system is unstable.

3.) With 2 extra poles at the origin, every point on the Nyquist diagram corresponding to $0 < |\omega| < \infty$ is shifted by $\pm 180°$
So the Nyquist diagram now looks like:



with 3 poles at the origin, $\measuredangle G(j\omega)$ now changes by $-540°$ as ω goes from 0^- to 0^+.

Now we have $N=2$ for all values of positive gain
$\Rightarrow Z=2$
$\Rightarrow$ the CL system is now unstable for all $K>0$