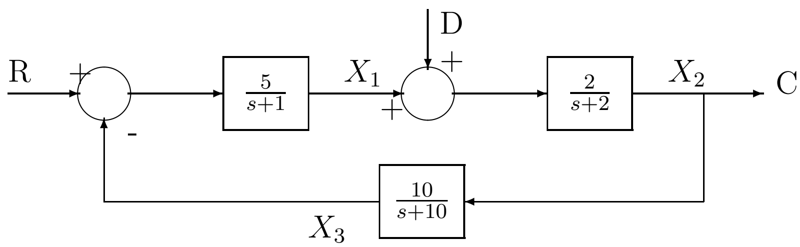
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Aeronautics and Astronautics
16.060: Principles of Automatic Control
Fall 2003

PROBLEM SET 5
Solutions

Problem 1

11.9(a) Define the states as shown below:



The input, output, and state vectors are:

$$\vec{u} = \begin{bmatrix} r \\ d \end{bmatrix} \qquad y = c \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

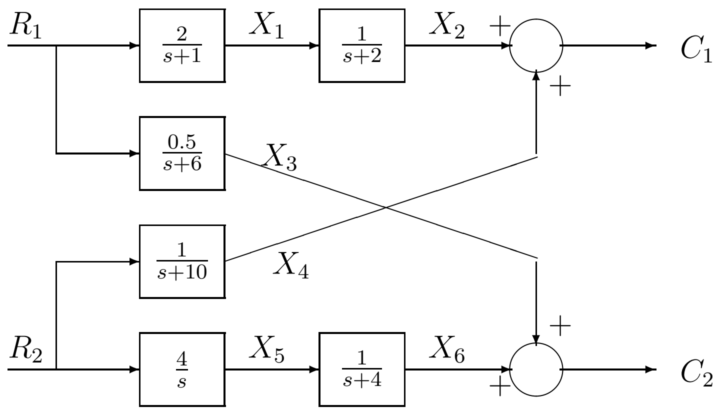
From the block diagram, we can write down the state equations and the output equation:

$$\begin{aligned} \dot{x_1} &= 5(r - x_3) - x_1 = -x_1 - 5x_3 + r \\ \dot{x_2} &= 2(x_1 + d) - 2x_2 = 2x_1 - 2x_2 + 2d \\ \dot{x_3} &= 10x_2 - 10x_3 \\ y &= x_2 \end{aligned}$$

In matrix form:

$$\begin{aligned} \dot{\vec{x}} &= \begin{bmatrix} -1 & 0 & -5 \\ 2 & -2 & 0 \\ 0 & 10 & -10 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \vec{u} \\ y &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 & 0 \end{bmatrix} \vec{u} \end{aligned}$$

11.9(b) Define the states as shown below:



The input, output, and state vectors are:

$$\vec{u} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

From the block diagram, we can write down the state equations and the output equation:

$$\begin{aligned}
\dot{x_1} &= 2r_1 - x_1 \\
\dot{x_2} &= x_1 - 2x_2 \\
\dot{x_3} &= -6x_3 + 0.5r_1 \\
\dot{x_4} &= -10x_4 + r_2 \\
\dot{x_5} &= 4r_2 \\
\dot{x_6} &= x_5 - 4x_6 \\
c_1 &= x_2 + x_4 \\
c_2 &= x_3 + x_6
\end{aligned}$$

In matrix form:

$$\begin{aligned}
\dot{\vec{x}} &= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 & 0 \\ 0 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{bmatrix} \vec{x} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0.5 & 0 \\ 0 & 1 \\ 0 & 4 \\ 0 & 0 \end{bmatrix} \vec{u} \\
\vec{y} &= \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{u}
\end{aligned}$$

Problem 2

1.

$$\begin{aligned} sI - A &= \begin{bmatrix} s+1 & 0 \\ -1 & s+2 \end{bmatrix} \\ \Rightarrow (sI-A)^{-1} &= \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+2 & 0 \\ 1 & s+1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{s+2} \end{bmatrix} \\ \Rightarrow \Phi(t) &= \mathcal{L}^{-1}[(sI-A)^{-1}] = \begin{bmatrix} e^{-t} & 0 \\ e^{-t} - e^{-2t} & e^{-2t} \end{bmatrix} \end{aligned}$$

2.

$$\Phi(0) = \begin{bmatrix} e^{-0} & 0 \\ e^{-0} - e^{-2(0)} & e^{-2(0)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3.

$$\begin{aligned} \Phi^{-1}(t) &= \frac{1}{e^{-3t}} \begin{bmatrix} e^{-2t} & 0 \\ e^{-2t} - e^{-t} & e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} e^{t} & 0 \\ e^{t} - e^{2t} & e^{2t} \end{bmatrix} \\ &= \Phi(-t) \end{aligned}$$

4.

$$\begin{aligned} \Phi(t_2)\Phi(t_1) &= \begin{bmatrix} e^{-t_2} & 0 \\ e^{-t_2} - e^{-2t_2} & e^{-2t_2} \end{bmatrix} \begin{bmatrix} e^{-t_1} & 0 \\ e^{-t_1} - e^{-2t_1} & e^{-2t_1} \end{bmatrix} \\ &= \begin{bmatrix} e^{-t_1-t_2} & 0 \\ [e^{-t_1-t_2} - e^{-t_1+2t_2} + e^{-t_1+2t_2} - e^{-2t_1-2t_2}] & e^{-2t_1-2t_2} \end{bmatrix} \\ &= \begin{bmatrix} e^{-(t_1+t_2)} & 0 \\ e^{-(t_1+t_2)} - e^{-2(t_1+t_2)} & e^{-2(t_1+t_2)} \end{bmatrix} \\ &= \Phi(t_1+t_2) \end{aligned}$$

Problem 3

1. (a) The characteristic equation for the matrix A is:

$$\det(\lambda I - A) = 0 \Longleftrightarrow \lambda^2 + 4\lambda + 4 = 0$$

So the system has repeated eigenvalues at $\lambda = -2$. Since these eigenvalues lie in the left-half plane, the system is stable.

(b)

$$\begin{aligned} sI - A &= \begin{bmatrix} s+1 & -1 \\ 1 & s+3 \end{bmatrix} \\ \Rightarrow (sI-A)^{-1} &= \frac{1}{(s+2)^2} \begin{bmatrix} s+3 & 1 \\ -1 & s+1 \end{bmatrix} = \begin{bmatrix} \frac{1}{(s+2)^2} + \frac{1}{s+2} & \frac{1}{(s+2)^2} \\ \frac{-1}{(s+2)^2} & \frac{-1}{(s+2)^2} + \frac{1}{s+2} \end{bmatrix} \\ \Rightarrow \Phi(t) &= \begin{bmatrix} (1+t)e^{-2t} & te^{-2t} \\ -te^{-2t} & (1-t)e^{-2t} \end{bmatrix} \end{aligned}$$

(c)

$$\vec{x}(t) = \Phi(t)\vec{x}(0) = \begin{bmatrix} (1+t)e^{-2t} & te^{-2t} \\ -te^{-2t} & (1-t)e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}$$

2. First find $\Phi(t)$:

$$\begin{aligned} sI - A &= \begin{bmatrix} s+1 & -1 \\ 0s+2 & \end{bmatrix} \\ \Rightarrow (sI-A)^{-1} &= \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+2 & 1 \\ 0 & s+1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} - \frac{1}{s+2} \\ 0 & \frac{1}{s+2} \end{bmatrix} \\ \Rightarrow \Phi(t) &= \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \end{aligned}$$

We are given the initial conditions $\vec{x}(0)$ and the input $u(t)$, so we can calculate the state vector at any time $t > 0$:

$$\begin{aligned} \vec{x}(t) &= \Phi(t)\vec{x}(0) + \int_0^t \Phi(t-\tau)B\vec{u}(\tau)d\tau \\ &= \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ 0 & e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} (1)d\tau \\ &= \begin{bmatrix} 2e^{-t} - e^{-2t} \\ e^{-2t} \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-\tau)} - e^{-2(t-\tau)} \\ e^{-2(t-\tau)} \end{bmatrix} d\tau \\ &= \begin{bmatrix} 2e^{-t} - e^{-2t} \\ e^{-2t} \end{bmatrix} + \begin{bmatrix} e^{-(t-\tau)} - \frac{1}{2}e^{-2(t-\tau)} \\ \frac{1}{2}e^{-2(t-\tau)} \end{bmatrix}_{\tau=0}^{\tau=t} \\ &= \begin{bmatrix} 2e^{-t} - e^{-2t} \\ e^{-2t} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ \frac{1}{2} - \frac{1}{2}e^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}(1 + 2e^{-t} - e^{-2t}) \\ \frac{1}{2}(1 + e^{-2t}) \end{bmatrix} \end{aligned}$$

Now that we have $\vec{x}(t)$, we can calculate $\vec{y}(t) = C\vec{x}(t)$:

$$\vec{y}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(1 + 2e^{-t} - e^{-2t}) \\ \frac{1}{2}(1 + e^{-2t}) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1 + 2e^{-t} - e^{-2t}) \\ \frac{1}{2}(1 + e^{-2t}) \end{bmatrix}$$

Problem 4

1. When a system is uncontrollable, it means that there are points in the state-space (i.e. sets of values of the given state variables) that can not be achieved no matter what input is used. A very simple example is an aircraft flying when there is a wind. The pilot can achieve any groundspeed by pointing the aircraft in the right direction and setting the throttle appropriately, and he can also achieve any airspeed by doing the same. But the pilot doesn't have the control he needs to set groundspeed and airspeed independently.

2. (a)

$$\Gamma = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & -6 \end{bmatrix}$$

The matrix has 2 independent columns, so the system is controllable.

(b)

$$\Gamma = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The matrix has 2 independent columns, so the system is controllable.

(c)

$$\Gamma = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & -3 \\ 1 & -3 & 5 \end{bmatrix}$$

The matrix has 3 independent columns (check: $\det \Gamma \neq 0$), so the system is controllable.

(d)

$$\Gamma = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

The matrix must have fewer than 3 independent columns, because the first row is all zeros, so the system is uncontrollable.