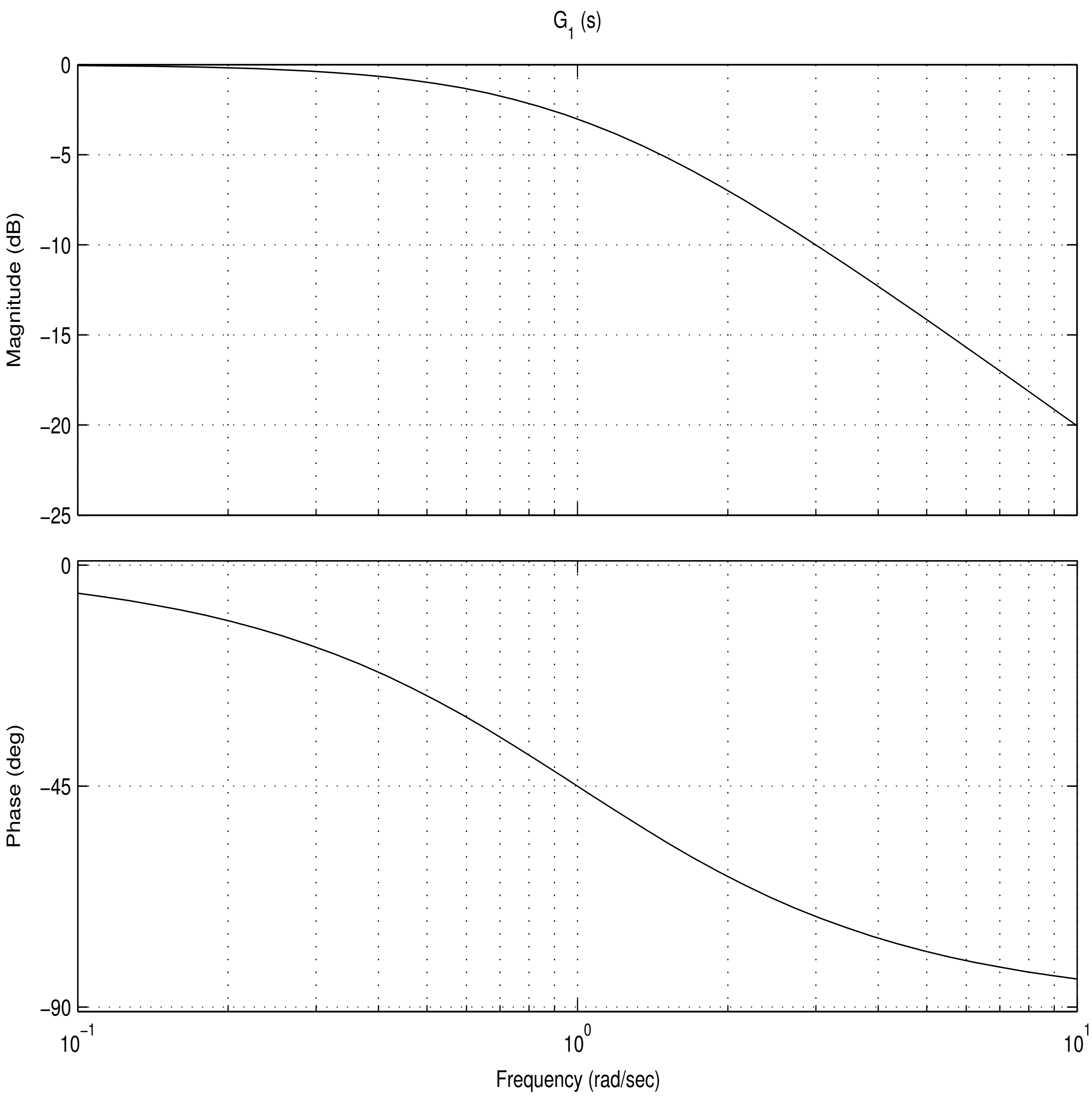
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Aeronautics and Astronautics
16.060: Principles of Automatic Control
Fall 2003

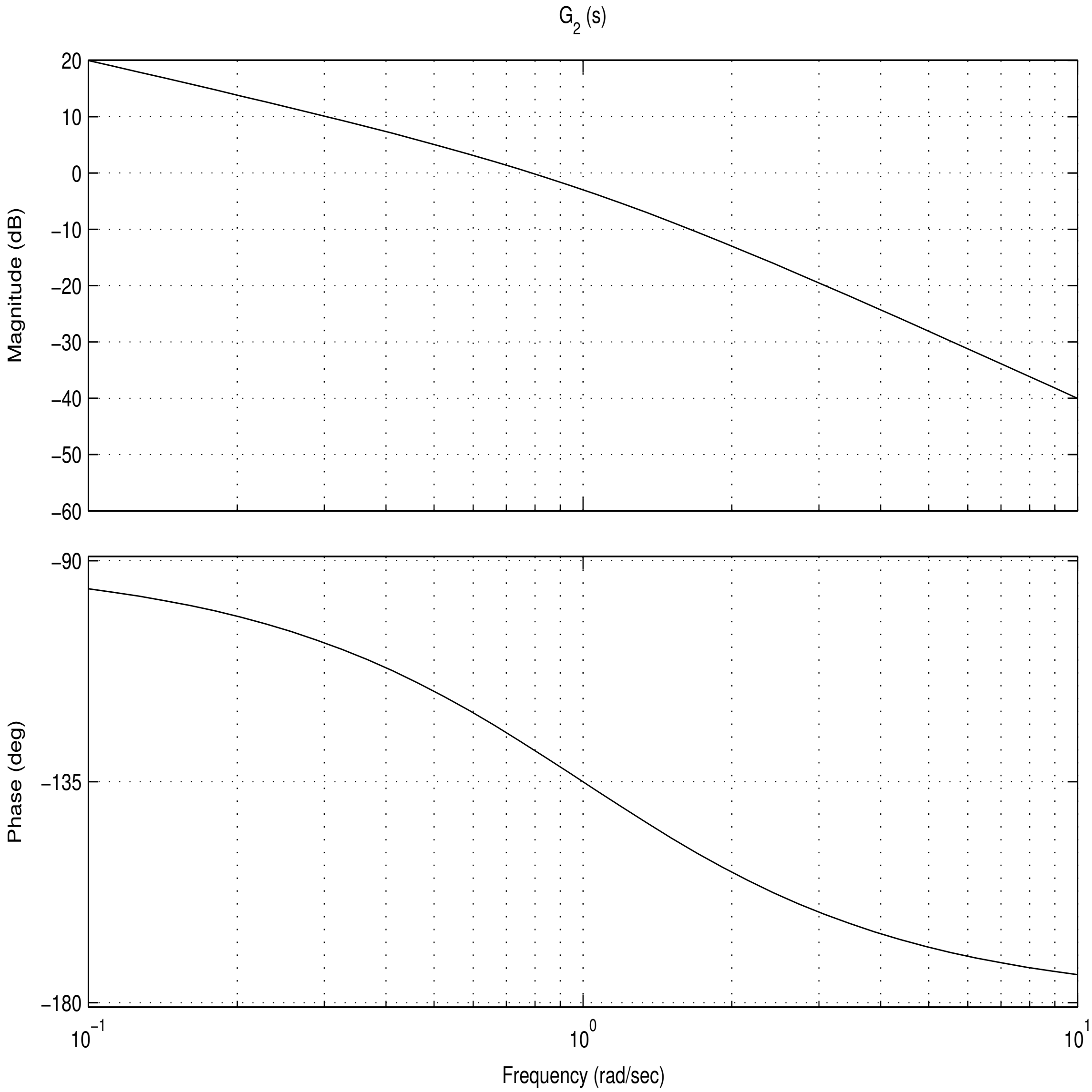
PROBLEM SET 10
Solutions

Problem 1

1.
   - The Bode gain is $K_{Bode} = 1$, and there are no integrators, so the low-frequency asymptote is $M_{dB} = 0$. Again because there are no integrators, the phase at low frequencies is $0°$.
   - The break frequency of the pole is $\omega = 1$. At this frequency, the slope of the magnitude plot changes by -20 dB/dec. The phase changes by $-5°$ one decade before the break frequency (at $\omega = 0.1$), by $-45°$ at the break frequency (at $\omega = 1$), and by $-85°$ one decade after the break frequency (at $\omega = 10$). At high frequencies, the phase approaches $-90°$.

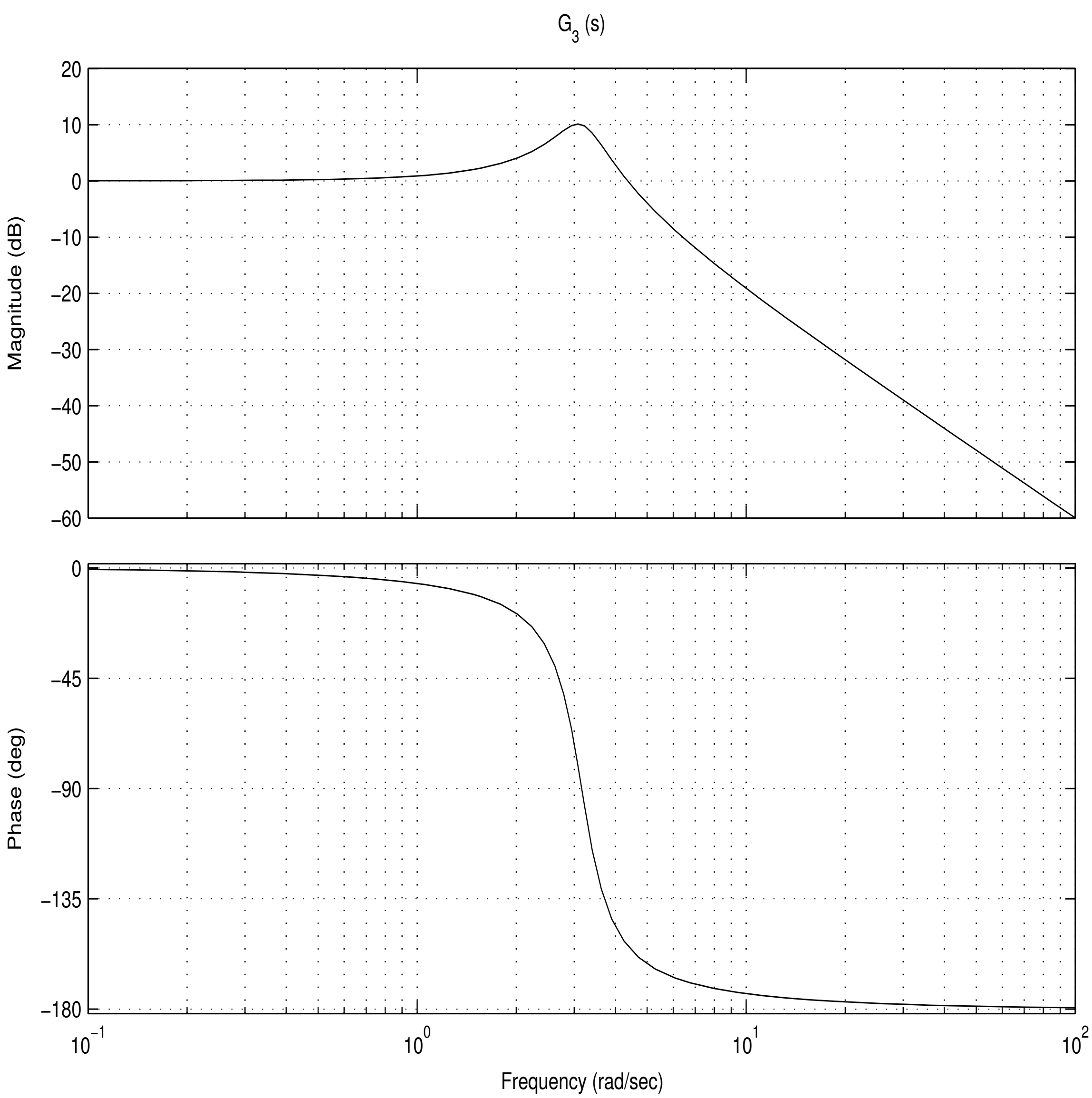


2.
- $K_{Bode} = 1$ and there is one integrator, so the low-frequency asymptote has a slope of -20 dB/dec and a magnitude of 1 (= 0 dB) at $\omega = 1$. Because there is one integrator, the phase at low frequencies is $-90°$.
- The pole at $s = -1$ breaks at $\omega = 1$, so the slope of the magnitude plot changes by -20 dB/dec at that point. (It goes from -20 dB/dec to -40 dB/dec.) The phase is approximately $-95°$ at $\omega = 0.1$, $-135°$ at $\omega = 1$, and $-175°$ at $\omega = 10$. The phase at high frequencies approaches $-180°$.

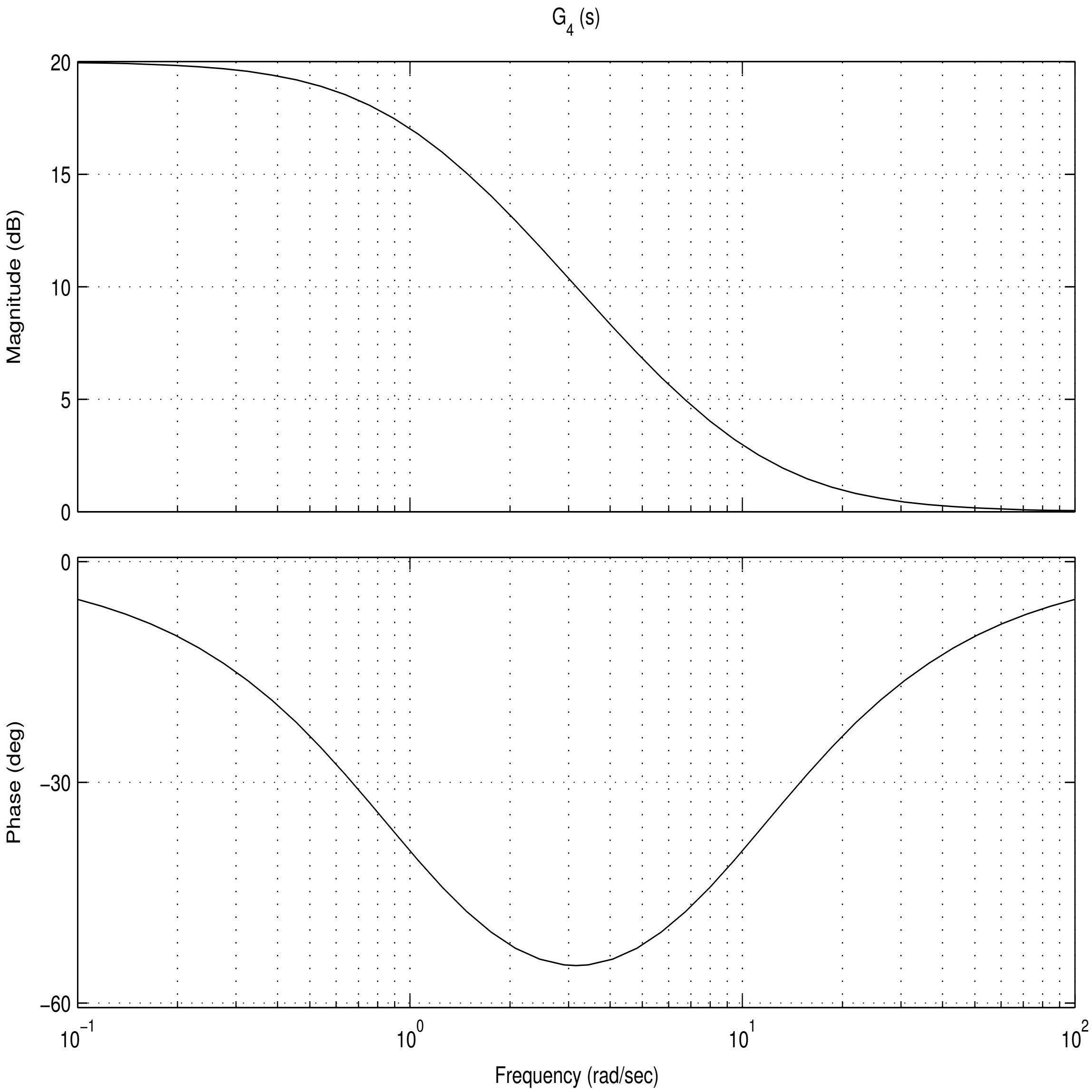


3. • $K_{Bode} = 1$ and there are no integrators, so the low-frequency asymptote is $M_{dB} = 0$ and the phase at low frequencies is zero.

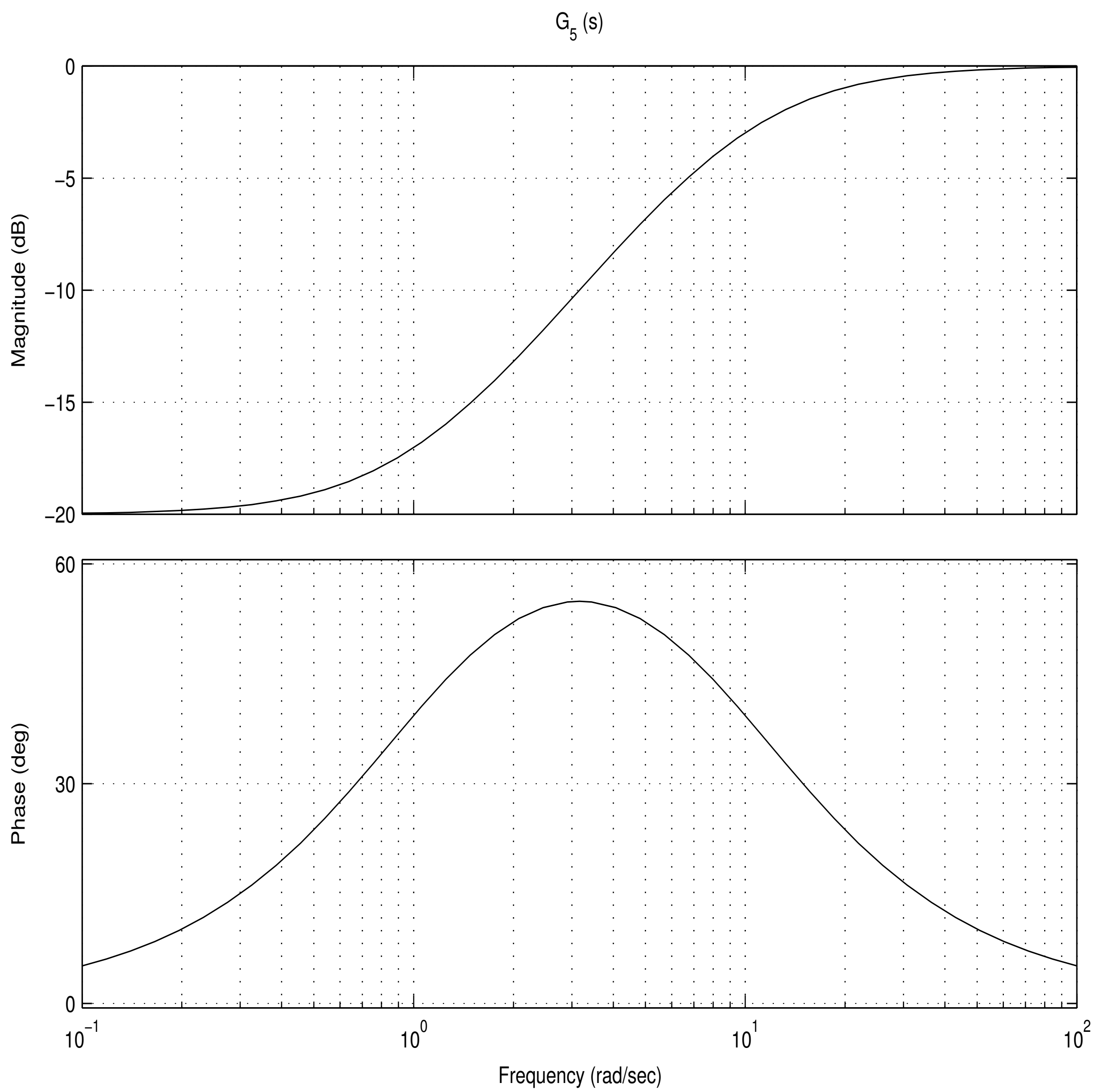
- There is a complex conjugate set of poles given by $s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + s + 10$. So $\omega_n = \sqrt{10}$ and $\zeta = \frac{1}{2\sqrt{10}} \approx 0.16$.
- The break frequency for the complex conjugate set of poles is at $\omega = \omega_n = \sqrt{10}$ (half-way between $\omega = 1$ and $\omega = 10$ on a log scale). At this frequency, the slope of the magnitude plot changes by -40 dB/dec. The phase changes by $-180°$ between approximately one decade before the break frequency ($\omega = 1/\sqrt{10}$) and one decade after ($\omega = 10\sqrt{10}$). (The steepness of the phase change depends on the value of ζ, but this approximation is good enough for this class.)
- From Figure 7.13 in Van De Vegte, a damping ratio of 0.16 corresponds to a peak in the magnitude plot of approximately 10 dB at $\omega = \omega_n$. The plot below (from Matlab) shows this peak.



4.
- $K_{Bode} = 10$ and there are no integrators, so the low-frequency asymptote is $M_{dB} = 20$ and the phase at low frequencies is zero.
- The first break frequency is due to the pole at $s = 1$. So at $\omega = 1$ the slope of the magnitude plot changes from 0 dB/dec to -20 dB/dec. Then the break frequency due to the zero is at $\omega = 10$, so at that frequency the slope changes from -20 dB/dec back to 0 dB/dec.
- At $\omega = 0.1$, the pole contributes $-5°$ and the zero contributes nothing, so the phase is $-5°$.
- At $\omega = 1$, the pole contributes $-45°$ and the zero contributes $+5°$, so the phase is $-40°$.
- At $\omega = 10$, the pole contributes $-85°$ and the zero contributes $+45°$, so the phase is $-40°$.
- At $\omega = 100$, the pole contributes $-90°$ and the zero contributes $+85°$, so the phase is $-5°$.
- As $\omega \to \infty$, the phase contribution from the pole and zero cancel and so the phase is $0°$.

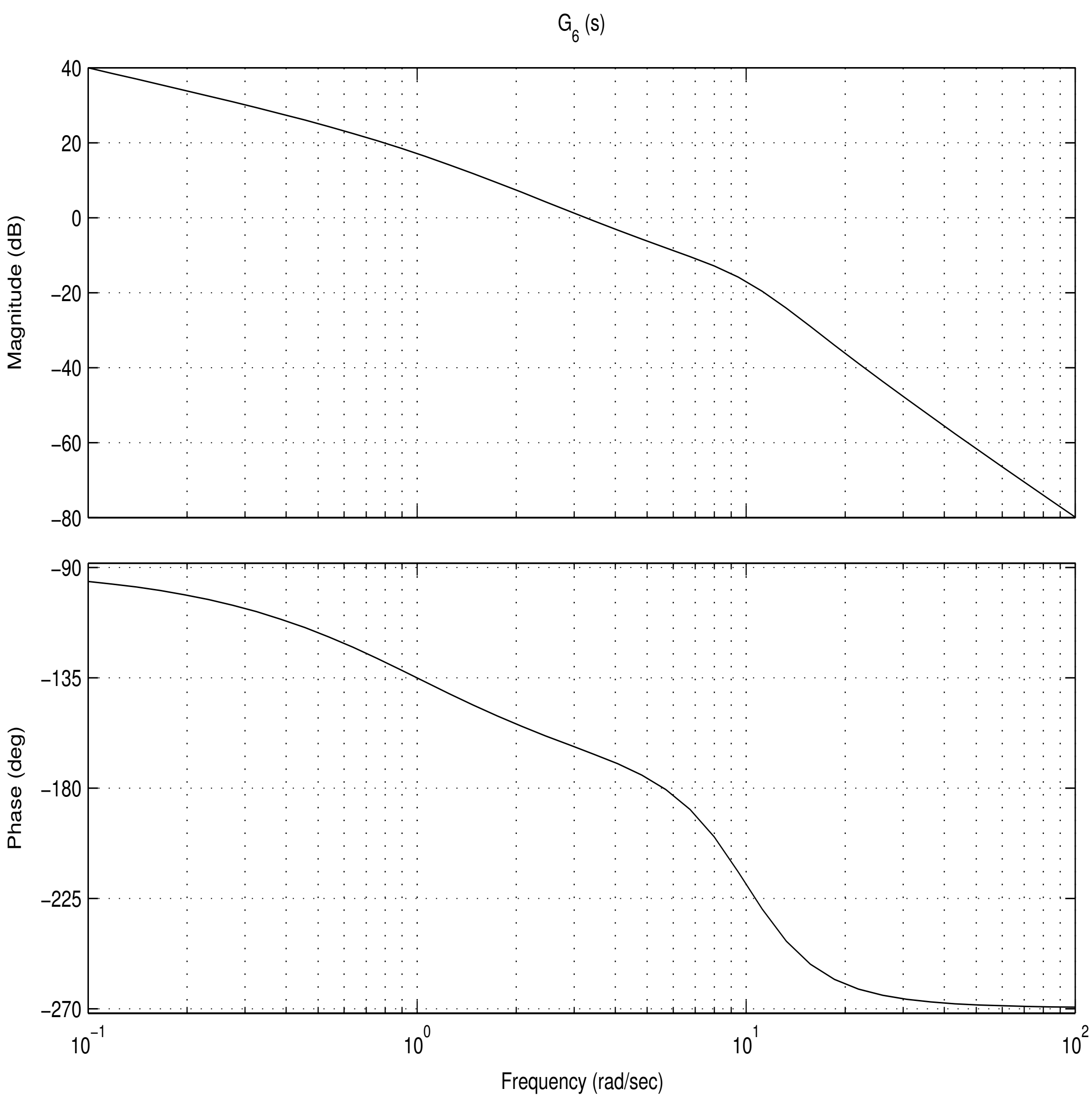


5. - $K_{Bode} = 0.1$ and there are no integrators, so the low-frequency asymptote is $M_{dB} = -20$ and the phase at low frequencies is zero.
   - The first break frequency is due to the zero at $s = 1$. So at $\omega = 1$ the slope of the magnitude plot changes from 0 dB/dec to +20 dB/dec. Then the break frequency due to the pole is at $\omega = 10$, so at that frequency the slope changes from +20 dB/dec back to 0 dB/dec.
   - At $\omega = 0.1$, the zero contributes $+5°$ and the pole contributes nothing, so the phase is $+5°$.
   - At $\omega = 1$, the zero contributes $45°$ and the pole contributes $-5°$, so the phase is $+40°$.
   - At $\omega = 10$, the zero contributes $85°$ and the pole contributes $-45°$, so the phase is $+40°$.
   - At $\omega = 100$, the zero contributes $90°$ and the pole contributes $-85°$, so the phase is $+5°$.
   - As $\omega \to \infty$, the phase contribution from the pole and zero cancel and so the phase is $0°$.

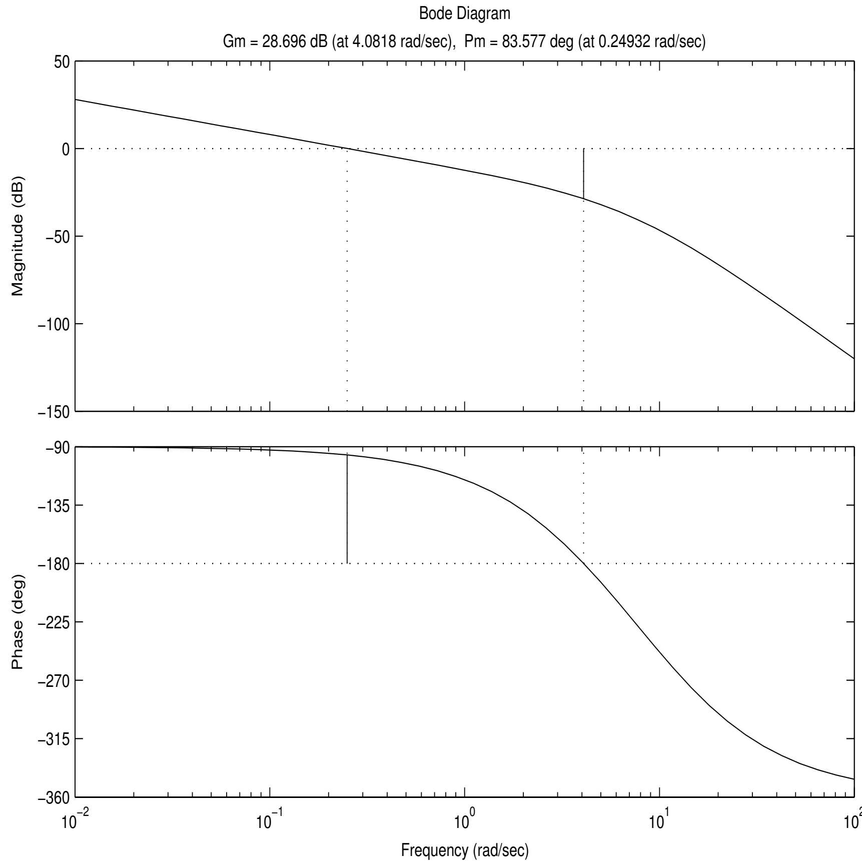


6.

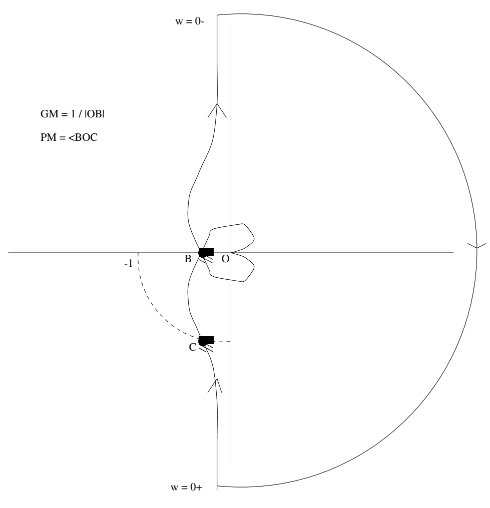
- $K_{Bode} = 10$ and there is one integrator, so the low-frequency asymptote has a slope of -20 dB/dec and a magnitude of 20 dB at $\omega = 1$. Because there is one integrator, the phase at low frequencies is $-90°$.
- The complex conjugate poles have $\omega_n = \sqrt{100} = 10$ and $\zeta = 0.5$.
- The pole at $s = -1$ breaks at $\omega = 1$, so the slope of the magnitude plot changes from -20 dB/dec to -40 dB/dec at that frequency. The pole contributes $-5°$ of phase shift at $\omega = 0.1$, $-45°$ at $\omega = 1$, and $-85°$ at $\omega = 10$.
- The zero at $s = -10$ and the complex conjugate poles both break at $\omega = 10$. The zero changes the slope by +20 dB/dec, and the complex poles change it by -40 dB/dec, so the net effect is a change in slope of -20 dB/dec. The slope therefore changes from -40 dB/dec to -60 dB/dec.
- The zero contributes $+5°$ of phase shift at $\omega = 1$, $+45°$ at $\omega = 10$, and $+85°$ at $\omega = 100$.
- The complex poles contribute approximately $-5°$ of phase shift at $\omega = 1$, $-90°$ at $\omega = 10$, and $-175°$ at $\omega = 100$. (If you wanted to plot the phase more accurately than that, you would use Matlab!)
- Because the damping ratio is so high, there is almost no peak in the magnitude at $\omega = \omega_n$.
- There are three more poles than zeros, so as $\omega \to \infty$, the phase approaches $-270°$.



Problem 2



Bode Diagram
Gm = 28.696 dB (at 4.0818 rad/sec), Pm = 83.577 deg (at 0.24932 rad/sec)
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)



w = 0-
GM = 1 / |OB|
PM = <BOC
-1
B
O
C
w = 0+

Problem 3

1. The first thing to notice is that the low-frequency asymptote is $M_{dB} = 20$ dB and the phase at low frequencies is zero, so there are no integrators and the Bode (standard) gain of $G_1(s)$ is 10. At around $\omega = 1$, the slope of the magnitude plot drops to -20 dB/dec and the phase turns negative, so there is a pole at $s = -1$. Then at around $\omega = 30$, the slope of the magnitude plot reverts to zero, and the phase starts getting more positive, so there is a zero at $s = -30$. Finally, at $\omega = 1000$, the slope drops to -20 dB/dec again and the phase gets more negative, so there is a pole at $s = -1000$. The final transfer function is then:

$$G_1(s) = \frac{10(\frac{s}{30} + 1)}{(s+1)(\frac{s}{1000} + 1)}$$

Check: there is one more pole than zero, so the phase at very high frequencies should be $-90°$ (it is).

2. Now the low-frequency asymptote has a slope of -20 dB/dec and a magnitude of 0 dB at $\omega = 1$, and the phase at very low frequencies is $-90°$. So there is one integrator, and the Bode gain is 1.

At $\omega = 1$, the slope changes to 0 dB/dec, indicating the presence of a zero. However, the phase changes by $-90°$, so the zero must be in the right half-plane at $s = 1$.

Then at $\omega = 100$, the slope changes to -20 dB/dec, indicating the presence of a pole. However, the phase changes by $+90°$ so the pole must be in the right half-plane at $s = 100$.

$$\Rightarrow G_2(s) = \frac{-s+1}{s(-\frac{s}{100} + 1)}$$

Problem 4

1. The crossover frequency is the frequency at which $|G(j\omega)| = 1$ or 0 dB. The gain margin is the difference in magnitude between 0 dB and $|G(j\omega)|$ when the phase is $-180°$. The phase margin is the difference in phase between $-180°$ and $\angle G(j\omega)$ when the magnitude is 1. From the tabulated data (or the Bode plot), we get:

$$\begin{aligned} \omega_c &\approx 3.1 \\ GM &\approx \frac{1}{0.25} = 4 \\ \phi_m &\approx 180° - 135° = 45° \end{aligned}$$

2. From Figure 8.7: a phase margin of 45° corresponds to a damping ratio of $\zeta \approx 0.43$. From Figure 8.6: this damping ratio corresponds to $\frac{\omega_c}{\omega_n} \approx 0.84$, so $\omega_n = \frac{3.1}{0.84} = 3.7$.

3. From Table 8.4.1 (or Figure 8.7): to get $\zeta = 0.6$, we need $\phi_m = 59°$.

4. To get this phase margin, we want the phase of $G(j\omega)$ to be $-180° + 59° = -121°$ when $|G(j\omega)| = 1$. From the tabulated data, $G(j\omega)$ has this phase at $\omega = 2.46$ and $|G(j\omega)| = 1.42$ at this frequency. So we need to multiply $G(j\omega)$ by a factor of $1/1.42 = .704$ to get the desired phase margin. So $K = .704$.

5. From Table 8.4.1: $\zeta = 0.6 \Rightarrow \frac{\omega_c}{\omega_n} = 0.72$. The new crossover frequency is 2.46, so the new natural frequency is $\omega_n = \frac{2.46}{0.72} = 3.4$.