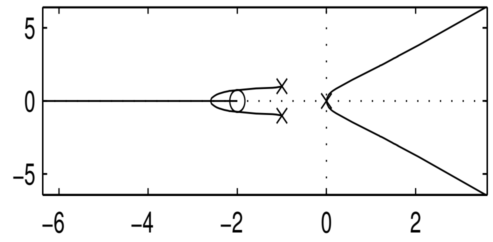
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Aeronautics and Astronautics
16.060: Principles of Automatic Control
Fall 2003

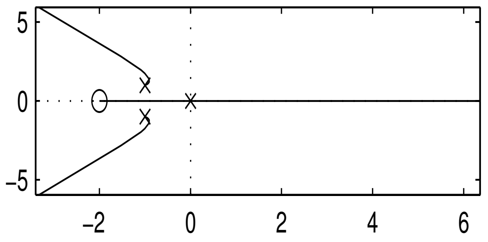
PROBLEM SET 7
Solutions

Problem 1

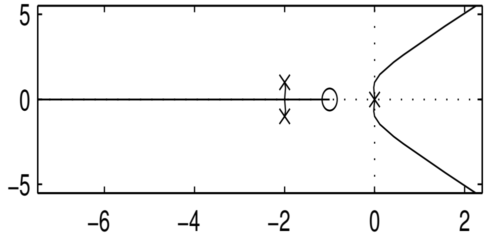
(1): K > 0



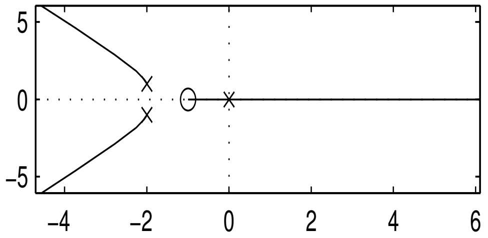
(1): K < 0



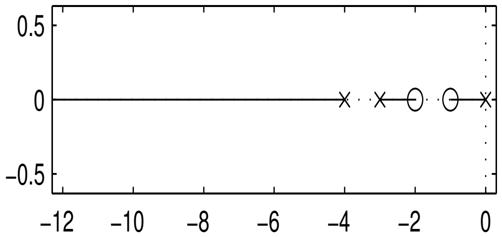
(2): K > 0



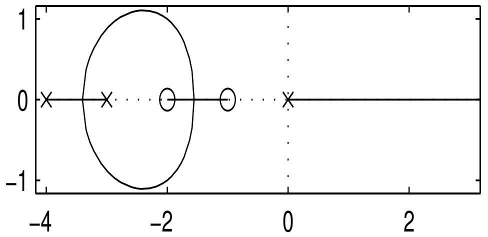
(2): K < 0



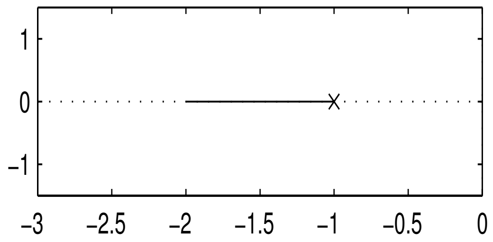
(3): K > 0



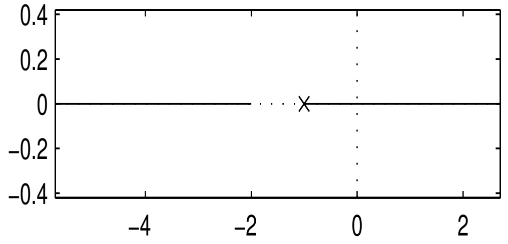
(3): K < 0



(4): K > 0

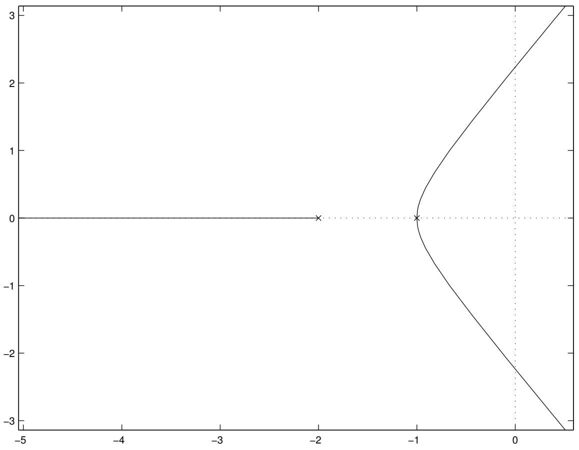


(4): K < 0



Problem 2

Problem 2.1



- 3 asymptotes with direction ±60°, 180°
- $\rho_0 = \frac{-1-1-2}{3} = -\frac{4}{3}$
- Let $\pm j\omega$ be the points of intersection of the root locus with the imaginary axis. To satisfy the angle condition, ω must satisfy the equation:

  $$-2\tan^{-1}(\omega) - \tan^{-1}(\frac{\omega}{2}) = -180^\circ$$

  By trial and error, we get $\omega \approx 2.2$. Now use the magnitude condition:

  $$K_{crit} = \sqrt{1^2 + 2.2^2} \cdot \sqrt{1^2 + 2.2^2} \cdot \sqrt{2^2 + 2.2^2} = 17.4$$

- We can also solve for K_{crit} algebraically. The characteristic equation is:

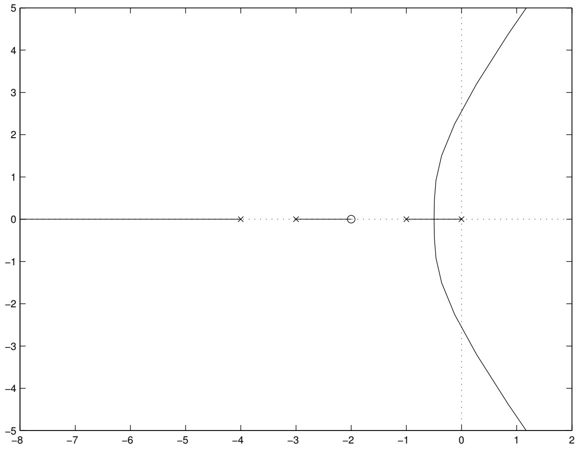
  $$\begin{aligned} (s+1)^2(s+2) + K &= 0 \\ s^3 + 4s^2 + 5s + 2 + K &= 0 \end{aligned}$$

  Now substitute $s = j\omega$ and $K = K_{crit}$:

  $$\begin{aligned} (j\omega)^3 + 4(j\omega)^2 + 5(j\omega) + 2 + K_{crit} &= 0 \\ (-4\omega^2 + 2 + K_{crit}) + j(-\omega^3 + 5\omega) &= 0 \end{aligned}$$

  Both real and imaginary parts of this equation must be zero. From the imaginary part, we get $\omega = \sqrt{5}$, and from the real part we get $K_{crit} = 4\omega^2 - 2 = 18$.

Problem 2.2



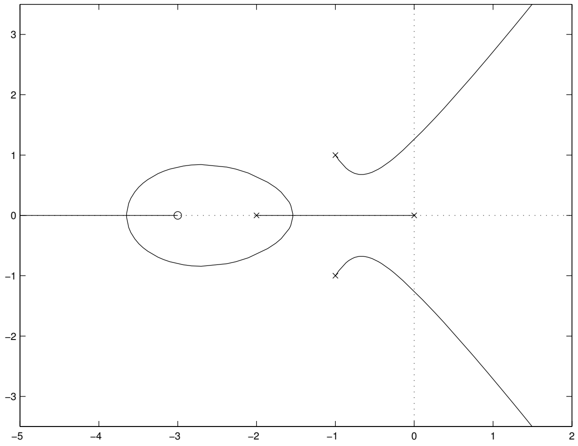
- 3 asymptotes with direction ±60°, 180°
- $\rho_0 = \frac{(0-1-3-4)-(-2)}{4-1} = -2$
- Let $\pm j\omega$ be the points of intersection of the root locus with the imaginary axis. To satisfy the angle condition, ω must satisfy the equation:

  $$\tan^{-1}(\frac{\omega}{2}) - 90^\circ - \tan^{-1}(\omega) - \tan^{-1}(\frac{\omega}{3}) - \tan^{-1}(\frac{\omega}{4}) = -180^\circ$$

By trial and error, we get $\omega \approx 2.6$. Now use the magnitude condition:

$$K_{crit} = \frac{2.6 \cdot \sqrt{1^2+2.6^2} \cdot \sqrt{3^2+2.6^2} \cdot \sqrt{4^2+2.6^2}}{\sqrt{2^2+2.6^2}} = 41.8$$

Problem 2.3



- 3 asymptotes with direction ±60°, 180°
- $\rho_0 = \frac{(0-1-1-2)-(-3)}{4-1} = -\frac{1}{3}$
- Let γ be the angle of departure from the upper complex pole. From the angle condition:

$$\begin{aligned} \tan^{-1}(\frac{1}{2}) - 135^\circ - 90^\circ - 45^\circ - \gamma &= -180^\circ \\ \Rightarrow \gamma = -63.4^\circ \end{aligned}$$

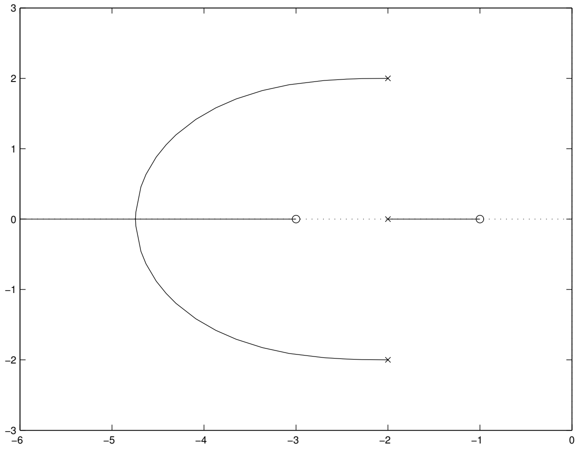
- Let $\pm j\omega$ be the points of intersection of the root locus with the imaginary axis. To satisfy the angle condition, ω must satisfy the equation:

$$\tan^{-1}(\frac{\omega}{3}) - 90^\circ - \tan^{-1}(\frac{\omega}{2}) - \tan^{-1}(\frac{\omega-1}{1}) - \tan^{-1}(\frac{\omega+1}{1}) = -180^\circ$$

By trial and error, we get $\omega \approx 1.25$. Now use the magnitude condition:

$$K_{crit} = \frac{1.25 \cdot \sqrt{2^2+1.25^2} \cdot \sqrt{1^2+0.25^2} \cdot \sqrt{1^2+2.25^2}}{\sqrt{3^2+1.25^2}} = 2.3$$

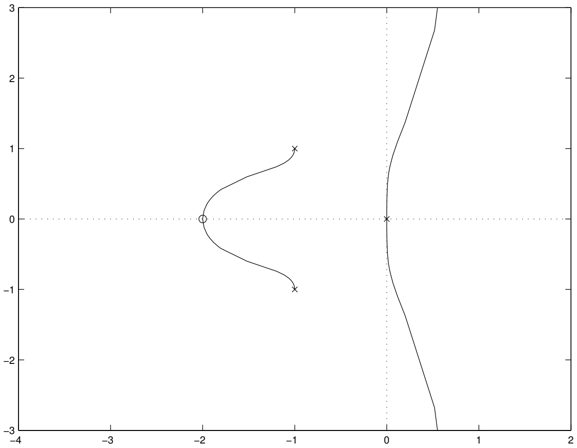
Problem 2.4



- 1 asymptote with direction 180°
- ρ_0 is not really meaningful in this case, because the asymptote is parallel with the real axis.
- Let γ be the angle of departure from the upper complex pole. From the angle condition:

$$\begin{aligned} 135^\circ + 45^\circ - 90^\circ - 90^\circ - \gamma &= -180^\circ \\ \Rightarrow \gamma = 180^\circ \end{aligned}$$

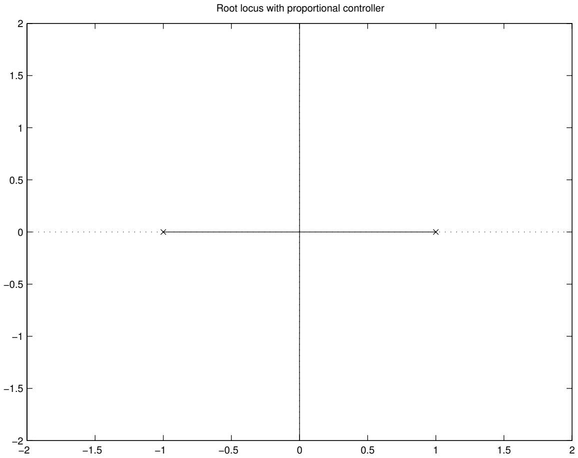
Problem 2.5



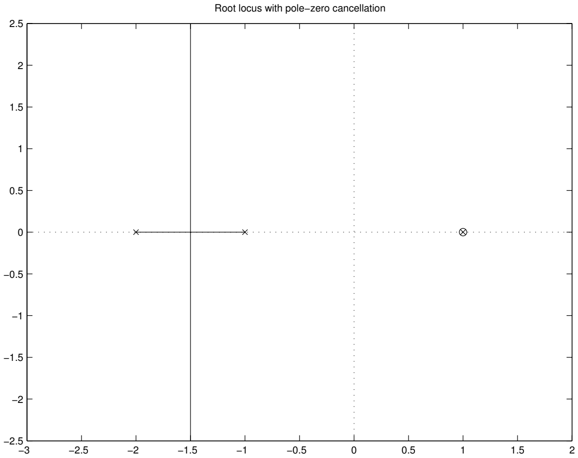
- 2 asymptotes with direction ±90°
- $\rho_0 = \frac{(0+0-1-1)-(-2-2)}{4-2} = 1$
- Let γ be the angle of departure from the upper complex pole. From the angle condition:

$$\begin{aligned} 45^\circ + 45^\circ - 135^\circ - 135^\circ - 90^\circ - \gamma &= -180^\circ \\ &\Rightarrow \gamma = -90^\circ \end{aligned}$$

Problem 3

1. 

As the root locus plot on the left shows, it's impossible to stabilize this system with only a proportional controller. The best that can be achieved is to place the closed-loop poles on the imaginary axis, but then the system is only marginally stable.

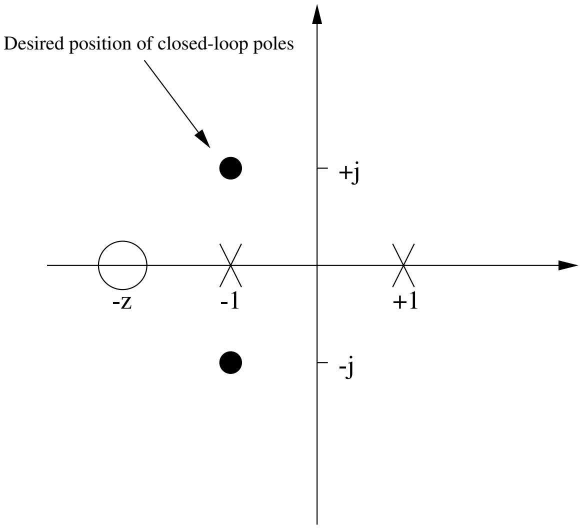
2. 

In theory, this controller would work because the zero at $s = 1$ cancels the unstable pole. But in practice, we can never model the position of the pole exactly, and its position will move around to a certain extent because of parameter variations in the plant, so perfect pole-zero cancellation is unrealistic. Without perfect cancellation, the mode due to the pole at $s = 1$ will still exist, and the system will still be unstable.

3. Note that adding a PD controller is equivalent to adding a zero. Let the position of the zero be at $s = -z$, and let K_c be the root locus gain of the controller:

$$G_c(s) = K_p + K_d s = K_c(s + z)$$

We want the closed-loop poles to have $\zeta \approx 0.7$ and $T_s = \frac{4}{\zeta\omega_n} = 4$. So the closed-loop poles should be at approximately $s = -1 \pm j$.



Now use the angle condition to find the position of the zero such that the root locus will pass through the point $s = -1 + j$:

$$\tan^{-1}(\frac{1}{z-1}) - 90^\circ - (180^\circ - \tan^{-1}(\frac{1}{2})) = -180^\circ \quad \Rightarrow \quad z = 1.5$$

Now that we have the position of the zero, use the magnitude condition to find the value of the root locus gain that will put the closed-loop poles at the desired positions:

$$K_{RL} = \frac{(1)(\sqrt{2^2+1^2})}{\sqrt{0.5^2+1^2}} = 2$$

Now the root locus gain of the system is just the root locus gain of the controller times the root locus gain of the plant $\Rightarrow K_{RL} = K_c K$. But we are given that $K = 1$, so the controller root locus gain must be $K_c = 2$. So the controller we should use is:

$$G_c(s) = 2(s + 1.5) = 3 + 2s$$