MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Aeronautics and Astronautics
16.060: Principles of Automatic Control
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PROBLEM SET 3
Solutions

Problem 1

1.

$$\begin{aligned} G(s) &= \frac{10(s+2)}{s+5} \cdot \frac{s+4}{4} \cdot \frac{10}{s+10} \\ &= 25\frac{(s+2)(s+4)}{(s+5)(s+10)} \\ \Rightarrow K_{rl} &= 25 \end{aligned}$$

2.

$$\begin{aligned} G(s) &= \frac{10(0.25s+1)}{0.1s+1} \cdot \frac{2(0.5s+1)}{5(0.2s+1)} \\ &= 4\frac{(0.5s+1)(0.25s+1)}{(0.1s+1)(0.2s+1)} \\ \Rightarrow K &= 4 \end{aligned}$$

3. There are no integrators, so the type number of the system is 0. For a type 0 system with unity feedback, the steady-state error for a unit step input is given by $\frac{1}{1+K}$, where K is the "standard" gain of the transfer function $G(s)$. So the steady state error would be $\frac{1}{1+4} = 0.2$.

Problem 2

1. First write $C(s)$ in root locus form, with $R(s) = \frac{1}{s}$:

$$\begin{aligned} C(s) &= \frac{3}{2} \cdot \frac{s^2+5s+6}{6} \cdot \frac{4}{s^2+5s+4} \cdot \frac{1}{s} \\ &= \frac{(s+3)(s+2)}{s(s+4)(s+1)} \end{aligned}$$

So $K_{rl} = 1$. The pole-zero plot has zeros at $s = -2$ and $s = -3$, and poles at $s = 0$, $s = -1$, and $s = -4$. The partial fraction expansion is:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+4}$$

The residues are:

$$
\begin{aligned}
K_1 &= \frac{(2\angle 0)(3\angle 0)}{(1\angle 0)(4\angle 0)} = \frac{3}{2}\angle 0 = \frac{3}{2} \\
K_2 &= \frac{(1\angle 0)(2\angle 0)}{(1\angle 180)(3\angle 0)} = \frac{2}{3}\angle{-180} = -\frac{2}{3} \\
K_3 &= \frac{(2\angle 180)(1\angle 180)}{(4\angle 180)(3\angle 180)} = \frac{1}{6}\angle 0 = \frac{1}{6} \\
\Rightarrow C(s) &= \frac{3/2}{s} - \frac{2/3}{s+1} + \frac{1/6}{s+4}
\end{aligned}
$$

Taking the inverse Laplace transform gives the output as a function of time (for $t > 0$):

$$c(t) = \frac{3}{2} - \frac{2}{3}e^{-t} + \frac{1}{6}e^{-4t}$$

2. Using the same method as before:

$$C(s) = \frac{s+1}{s^2-25}\cdot 1 = \frac{s+1}{(s+5)(s-5)}$$

$K_{rl} = 1$ and there is a zero at $s = -1$ and two poles at $s = -5$ and $s = 5$.

$$
\begin{aligned}
C(s) &= \frac{K_1}{s+5} + \frac{K_2}{s-5} \\
K_1 &= \frac{4\angle 180}{10\angle 180} = \frac{2}{5}\angle 0 = \frac{2}{5} \\
K_2 &= \frac{6\angle 0}{10\angle 0} = \frac{3}{5}\angle 0 = \frac{3}{5}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow C(s) &= \frac{2/5}{s+5} + \frac{3/5}{s-5} \\
\Rightarrow c(t) &= \frac{2}{5}e^{-5t} + \frac{3}{5}e^{5t}
\end{aligned}
$$

Note that the output is unbounded, because of the pole in the right-half plane.

3. $$C(s) = \frac{2s+4}{s^2+2s+2}\cdot\frac{1}{s} = \frac{2(s+2)}{s(s^2+2s+2)}$$

$K_{rl} = 2$ and there is a zero at $s = -2$, and a complex conjugate pair of poles at $s = -1 \pm j$.

$$
\begin{aligned}
C(s) &= \frac{K_1}{s} + \frac{K_2}{s+1-j} + \frac{K_3}{s+1+j} \\
K_1 &= \frac{2(2\angle 0)}{(\sqrt{2}\angle{-45})(\sqrt{2}\angle 45)} = 2\angle 0 = 2 \\
K_2 &= \frac{2(\sqrt{2}\angle 45)}{(\sqrt{2}\angle 135)(2\angle 90)} = 1\angle{-180} = -1
\end{aligned}
$$

K_3 is just the complex conjugate of K_2, so $K_3 = -1$. To find $c(t)$, use the expression from Lecture 7 to calculate the term due to the pair of complex poles:

$$2Ke^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}\,t+\theta) \tag{1}$$

In this case, $K = |K_2| = 1$, $-\zeta\omega_n = -1$, $\omega_n\sqrt{1-\zeta^2} = 1$, and $\theta = \arg(K_2) = -180°$. So the output time response is given by:

$$c(t) = 2 + 2e^{-t}\cos(t - 180°) = 2 - 2e^{-t}\cos(t)$$

4. The input now is $R(s) = \mathcal{L}[e^{-5t}] = \frac{1}{s+5}$, so:

$$C(s) = \frac{4(s+2)}{(s+1)(s+5)(s^2+6s+18)}$$

$K_{rl} = 4$ and there is a zero at $s = -2$, and poles at $s = -1$, $s = -5$, and $s = -3 \pm 3j$.

$$\begin{aligned}
C(s) &= \frac{K_1}{s+1} + \frac{K_2}{s+5} + \frac{K_3}{s+3-3j} + \frac{K_4}{s+3+3j} \\
K_1 &= \frac{4(1\angle 0)}{(4\angle 0)(\sqrt{13}\angle{-56.3})(\sqrt{13}\angle 56.3)} = \frac{1}{13}\angle 0 = 0.0769 \\
K_2 &= \frac{4(3\angle 180)}{(4\angle 180)(\sqrt{13}\angle 236.3)(\sqrt{13}\angle 123.7)} = \frac{1}{13}\angle{-360} = 0.2308 \\
K_3 &= \frac{4(\sqrt{10}\angle 108.4)}{(\sqrt{13}\angle 123.7)(\sqrt{13}\angle 56.3)(6\angle 90)} = \frac{2\sqrt{10}}{39}\angle{-161.6} = 0.1622\ \angle{-161.6} \\
K_4 &= \overline{K_3} = 0.1622\ \angle 161.6 \\
\\
\Rightarrow c(t) &= 0.0769e^{-t} + 0.2308e^{-5t} + 0.3244e^{-3t}\cos(3t - 161.6°)
\end{aligned}$$

Problem 3

1. (a) There are two complex poles at $s = -3 \pm \sqrt{3}j$.

   (b)

$$\begin{aligned}
\zeta &= \cos 30° = \frac{\sqrt{3}}{2} = 0.866 \\
\omega_n &= \sqrt{3^2 + (\sqrt{3})^2} = \sqrt{12} = 3.464 \\
\omega_d &= \omega_n\sqrt{1-\zeta^2} = \sqrt{3} = 1.732 \\
T &= \frac{1}{\zeta\omega_n} = \frac{1}{3} = 0.333
\end{aligned}$$

   (c) The system is underdamped because there are poles that are off the real axis. The type number is zero, because there are no integrators, but the order is 2, because of the two poles.

2. (a) The steady-state output is 2, and the peak oputut is approximately 2.3, so the percent overshoot is $\frac{2.3-2}{2} = 15\%$. From Figure 5.2 in Van de Vegte:

$$\text{P.O.} = 15\% \Rightarrow \zeta \approx 0.52$$

The peak time is approximately 1.8, so from equation (5.4):

$$\omega_n = \frac{\pi}{T_p\sqrt{1-\zeta^2}} = 2.0$$

And now we can also estimate ω_d and T:

$$\begin{aligned}\omega_d &= \omega_n\sqrt{1-\zeta^2} = 1.7\\ T &= \frac{1}{\zeta\omega_n} = 0.96\end{aligned}$$

Since the steady-state value of the output is 2, and the input was a unit step, the gain must be $K = 2$.

(b) The poles are at $s = -\zeta\omega_n \pm \omega_d j \approx 1.04 \pm 1.7j$. With a gain of 2, the system transfer function must be:

$$G(s) = \frac{K{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2} = \frac{8}{s^2 + 2.08s + 4}$$

Problem 4

1.

$$\begin{aligned}G_1(s) &= \frac{0.5}{s+0.5}\\ G_2(s) &= \frac{8}{s^2+4s+8}\\ G_3(s) &= \frac{4}{(s+0.5)(s^2+4s+8)}\\ G_4(s) &= \frac{7.273(s+0.55)}{(s+0.5)(s^2+4s+8)}\end{aligned}$$

2. For G_1: $T = \frac{1}{0.5} = 2$.
   For G_2: $\zeta = \cos 45^\circ = \frac{1}{\sqrt{2}}$, $\omega_n = \sqrt{2^2+2^2} = 2\sqrt{2}$, $\omega_d = 2$, $T = \frac{1}{2}$.

3. (a)

$$\begin{aligned}C_1(s) &= \frac{K_1}{s} + \frac{K_2}{s+0.5}\\ K_1 &= \frac{0.5}{0.5\angle 0} = 1\\ K_2 &= \frac{0.5}{0.5\angle 180} = -1\\ \Rightarrow c_1(t) &= 1 - e^{-0.5t}\end{aligned}$$

(b)

$$
\begin{aligned}
C_2(s) &= \frac{K_1}{s} + \frac{K_2}{s+2-2j} + \frac{K_3}{s+2+2j} \\
K_1 &= \frac{8}{(2\sqrt{2}\angle{-45})(2\sqrt{2}\angle 45)} = 1 \\
K_2 &= \frac{8}{(2\sqrt{2}\angle 135)(4\angle 90)} = 0.707\angle 135 \\
\Rightarrow c_2(t) &= 1 + 1.41e^{-2t}\cos(2t + 135^\circ)
\end{aligned}
$$

(c)

$$
\begin{aligned}
C_3(s) &= \frac{K_1}{s} + \frac{K_2}{s+0.5} + \frac{K_3}{s+2-2j} + \frac{K_4}{s+2+2j} \\
K_1 &= \frac{4}{(0.5\angle 0)(2\sqrt{2}\angle{-45})(2\sqrt{2}\angle 45)} = 1 \\
K_2 &= \frac{4}{(0.5\angle 180)(\sqrt{1.5^2+2^2}\angle{-53.1})(\sqrt{1.5^2+2^2}\angle 53.1)} = -1.28 \\
K_3 &= \frac{4}{(2\sqrt{2}\angle 135)(\sqrt{1.5^2+2^2}\angle 126.9)(4\angle 90)} = 0.141\angle 8.1 \\
\Rightarrow c_3(t) &= 1 - 1.28e^{-0.5t} + 0.283e^{-2t}\cos(2t + 8.1^\circ)
\end{aligned}
$$

(d)

$$
\begin{aligned}
C_4(s) &= \frac{K_1}{s} + \frac{K_2}{s+0.5} + \frac{K_3}{s+2-2j} + \frac{K_4}{s+2+2j} \\
K_1 &= \frac{7.273(0.55\angle 0)}{(0.5\angle 0)(2\sqrt{2}\angle{-45})(2\sqrt{2}\angle 45)} = 1 \\
K_2 &= \frac{7.273(0.05\angle 0)}{(0.5\angle 180)(\sqrt{1.5^2+2^2}\angle{-53.1})(\sqrt{1.5^2+2^2}\angle 53.1)} = -0.116 \\
K_3 &= \frac{7.273(\sqrt{1.45^2+2^2}\angle 125.9)}{(2\sqrt{2}\angle 135)(\sqrt{1.5^2+2^2}\angle 126.9)(4\angle 90)} = 0.633\angle 134.0 \\
\Rightarrow c_4(t) &= 1 - 0.116e^{-0.5t} + 1.27e^{-2t}\cos(2t + 134^\circ)
\end{aligned}
$$

4. See Figure 1.
5. The dominant mode in (3) is clearly the simple lag at $s = -0.5$. It is dominant both in having a larger residue and a longer decay time. In (4), however, the zero at $s = -0.55$ almost cancels out the pole at $s = -0.5$, so it has a very small residue. Initially, therefore, the response is dominated by the quadratic lag because that mode has a much greater residue. However, the quadratic lag decays fairly quickly, and although the simple lag residue is small, it takes so long to decay that the output takes a long time to achieve its steady-state value, despite having very quick response early on.

Problem 5

1. (a) In the plant transfer function, the pole at $s = -1$ is much closer to the origin than the pole at $s = -4$, so the dominating time constant of the plant is $T = 1$.

   (b) First, find the closed-loop transfer function of the system:

$$
T(s) = \frac{KG(s)}{1 + KG(s)} = \frac{K}{s^2 + 5s + (K+4)}
$$

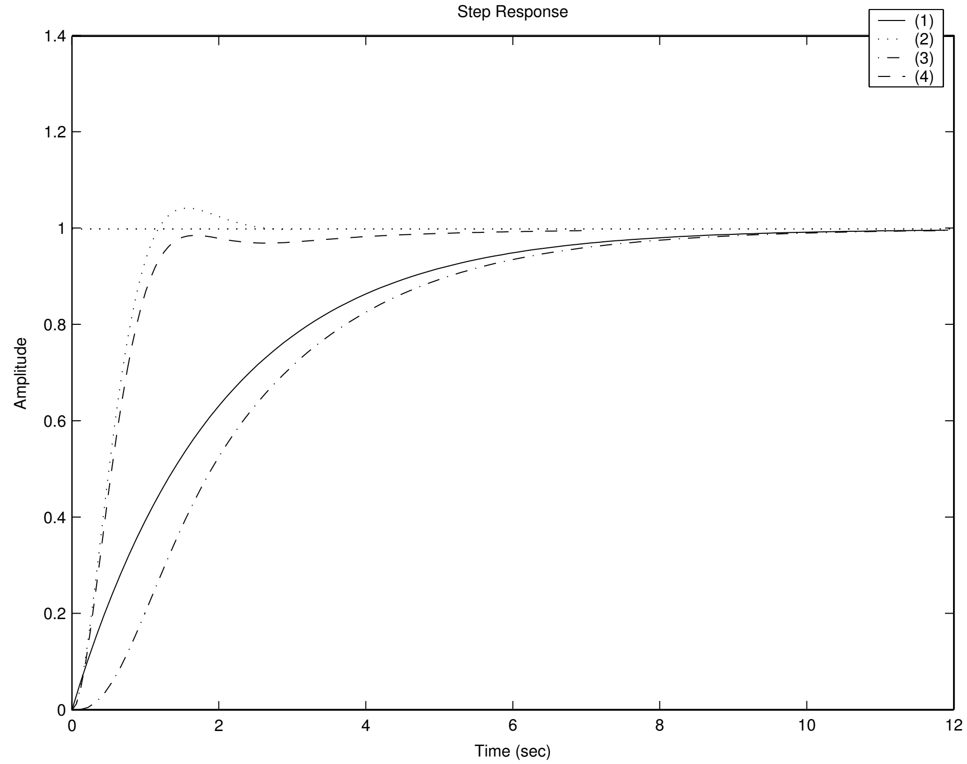

Figure 1: Step Responses (Problem 4.4)

The closed-loop poles are the just the roots of the characteristic equation of $T(s)$: $s^2 + 5s + (K+4) = 0$. To make the dominating time constant half as much as for the open-loop system, we need a closed-loop pole at $s = -2$, since that will yield a time constant of $\frac{1}{2}$. So set $s = -2$ in the characteristic equation and solve for K to get $K = 2$.

(c) With $K = 2$ and $R(s) = \frac{1}{s}$, our closed-loop output is given by:

$$C(s) = \frac{2}{s(s^2 + 5s + 6)}$$

Use graphical determination of residues to find $c(t)$:

$$\begin{aligned} C(s) &= \frac{K_1}{s} + \frac{K_2}{s+2} + \frac{K_3}{s+3} \\ K_1 &= \frac{2}{(2\angle 0)(3\angle 0)} = \frac{1}{3} \\ K_2 &= \frac{2}{(2\angle 180)(1\angle 0)} = -1 \\ K_3 &= \frac{2}{3\angle 180)(1\angle 180)} = \frac{2}{3} \\ \Rightarrow c(t) &= \frac{1}{3} - e^{-2t} + \frac{2}{3}e^{-3t} \end{aligned}$$

We can see that the steady-state value of $c(t)$ is $\frac{1}{3}$ in response to an input of 1, so there is a steady-state error of $\frac{2}{3}$. (You could also determine that by using the steady-state error table.)

(d) A higher value of K will reduce the steady-state error, but it will also cause the output to start oscillating. To understand why, look again at the characteristic equation: as K increases, the roots of the equation become complex, so the closed-loop poles move off the real axis. As K

increases more, the poles move further away from the real axis, and so the damping ratio goes down and the output gets more oscillatory. For a physical system, it is generally undesirable to have highly oscillatory behavior because components are more likely to break!

To find the value of K that will give a damping ratio of 0.7, equate the denominator of the closed-loop transfer function with the denominator of a generic quadratic lag system:

$$s^2 + 5s + (K+4) = s^2 + 2\zeta\omega_n s + {\omega_n}^2$$

So we have:

$$\begin{aligned} 2\zeta\omega_n &= 5 \\ {\omega_n}^2 &= K+4 \end{aligned}$$

Setting $\zeta = 0.7$ and solving for K, we get $K \approx 8.76$. To get the corresponding steady-state error, first find the gain of the open-loop system, which is the gain of the controller K multiplied by the gain of $G(s)$—that comes out to $(8.76) * (1/4) = 2.19$. Then we can use the steady-state error table: a type 0 system with a step input will have a steady-state error of $\frac{1}{1+gain} = \frac{1}{1+2.19} = 0.314$.

2. (a) To find the poles of $G(s)$, set the denominator of $G(s)$ to zero and solve for s. That gives poles at $s = -1 \pm 3.45j$. Because the poles are complex, the system is underdamped (i.e. the output will overshoot and oscillate before it settles to its steady-state value).

   (b) Using the same method as in Problem 5, Question 1(b), we get the characteristic equation of the closed-loop transfer function:

   $$s^2 + 2s + (12.9 + K) = 0$$

   Setting $K = -10$ and solving for s, we find the closed-loop poles to be at $s = -1 \pm 1.38j$. The system is more damped than before (since the poles have moved closer to the real axis), but it is still underdamped.

   (c) For $K = -12$, the poles are at $s = -0.684$ and $s = -1.32$. The system is now overdamped.

   (d) For $K = -20$, the poles are at $s = 1.85$ and $s = -3.85$. There is now a pole in the right-half plane, so the system is unstable! (Don't try flying this aircraft...)