16.06 Lecture 28

Principle of the Argument

November 12, 2003

Today's Topics:

1. D Contours
2. The Principle
3. Zeros of G(s)+1

An important and useful approach to control system design is obtained by considering closed D contours in the s plane. The D contour looks like this

Notice that the vertical portion of the contour travels up the imaginary axis, which we know yields the system frequency response in the $G(s)$ plane. For example, consider the first order system, which is a pole in the left half plane

The term $s-(-2)$ is the vector from the pole at -2 to the point s in the plane. In polar form the transfer function becomes

Thus, if we traverse the D contour in the s plane, it produces a $G(s)$ contour

Note that the $G(s)$ contour does <u>not</u> enclose the origin. In fact, if there are N poles in the left half plane then there will be N loops but none of them will enclose the origin.

Now consider a system with a pole in the right half plane (RHP)

so

Note that there is one counterclockwise encirclement of the origin by the $G(s)$ contour. If there are N poles in the right half plane then there will be N counterclockwise encirclements of the origin by the $G(s)$ contour.

If there is a zero in he left half plane so

then the D contour is simply shifted to the right by +2.

and the contour does not encircle the origin

Alternatively, if the zero is in the right half plane so

then the D contour is simply shifted to the left by –2

and there is one clockwise encirclement of the origin by the $G(s)$ contour. If there are N zeros of $G(s)$ in the right half plane then there will be N clockwise encirclements of the origin.

These encirclement properties of $G(s)$ contours are summarized by an important mathematical result proved by the famous French mathematician Cauchy. It is called the Principle of the Argument.

Principle of the Argument for $G(s)$-If the function $G(s)$ has Z zeros and P poles inside the clockwise contour D, then the associated $G(s)$ contour will encircle the origin of a total of N=Z-P times, in the clockwise direction.

Furthermore, if we enlarge the D contour to infinity in the right half plane then we will know N for all poles and zeros in the entire right half plane.

Now in many was this result tells us nothing about control system design that we don't already know if we have $G(s)$. In particular, we can readily factor the numerator and denominator of $G(s)$ to obtain its poles and zeros. Thus we immediately know Z and P and can calculate N.

So why is this useful?

Well, let's consider closed loop systems. If we close the loop around $G(s)$ with unity feedback we get the closed loop transfer function

To study this closed loop system we are interested in its poles and zeros. What are the poles and zeros of the closed loop system?

Is a pole of $G(s)$ a pole of $G_{CL}(s)$?

Is a zero of $G(s)$ a pole of $G_{CL}(s)$?

Is a pole of $G(s)+1$ a pole of $G_{CL}(s)$?

Is a zero of $G(s)+1$ a pole of $G_{CL}(s)$?

In summary, we have-

The zeros of $G(s)$ are the zeros of $G_{CL}(s)$

The zeros of $G(s)+1$ are the poles of $G_{CL}(s)$

Thus, we need to find out about the zeros of $G(s)+1$. In particular, in order to study the stability of the closed loop system, we are interested in whether or not there are zeros of $G(s)+1$ in the RHP, because any zeros of $G(s)+1$ in the RHP are Poles of $G_{CL}(s)$ in the RHP.

Earlier we studied contours of $G(s)$ as we traced out the D contour in the right half plane and were able to discern the net number of poles and zeros inside the contour. However, now we are interested in $G(s)+1$. But if we know the $G(s)$ contour then we can readily obtain the $G(s)+1$ contour. It is simply the $G(s)$ contour shifted to the right by +1.But this is the same as moving the origin one unit to the left.

Thus we have-

Principle of the Argument for $G(s)+1$-If the function $G(s)+1$ has Z zeros and P poles inside the clockwise contour D, then the associated $G(s)$ contour will encircle the –1 point a total of N=Z-P times, in the clockwise direction.

Hence, if we extend the D contour to infinity and plot the $G(s)$ contour, then the net clockwise encirclements of the –1 point yield the N, the excess of zeros over poles of $G(s)+1$ in the RHP. Since we know P (the number of poles of $G(s)$ in the RHP) then we know Z, the number of zeros of $G(s)+1$ in the RHP

$$Z=N+P$$

Which is also the number of poles of $G(s)+1$ in the RHP.

Finally, we have-

<u>Nyquist Stability Criterion</u>-A feedback control system is stable if and only if Z=N+P=0, where N is the number of clockwise encirclements of the –1 point and P is the number of poles of the loop transfer function inside the RHP (unstable poles of LTF)

<u>Example-</u>