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16.07 Dynamics
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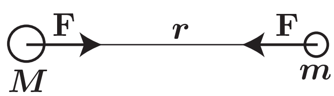
Lecture D15 - Gravitational Attraction. The Earth as a Non-Inertial Reference Frame

Gravitational attraction

The Law of Universal Attraction was already introduced in lecture D1. The law postulates that the force of attraction between any two particles, of masses M and m, respectively, has a magnitude, F, given by

$$F = G\frac{Mm}{r^2} \tag{1}$$

where r is the distance between the two particles, and G is the universal constant of gravitation. The value of G is empirically determined to be $6.673(10^(-11))m^3/(kg.s)^2$. The direction of the force is parallel to the line connecting the two particles.



Recall, from lecture D8, that the gravitational force is a conservative force that can be derived from a potential. The potential for the gravitation force is given by

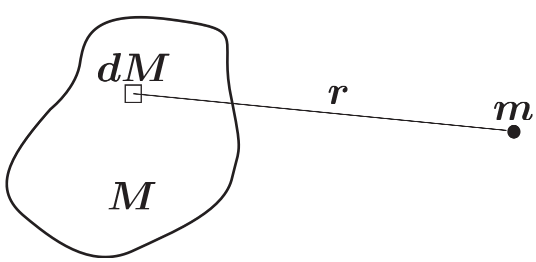
$$V = -G\frac{Mm}{r} \ ,$$

and

$$\boldsymbol{F} = -\boldsymbol{\nabla} V \ .$$

The law of gravitation stated above is strictly valid for point masses. One would expect that when the when the sizes of the masses are comparable to the distance between the masses, one would observe deviations to the above law. In such cases, the forces due to gravitational attraction would depend on the spatial distribution of the mass.

Consider the case in which the mass m has a small size and can be regarded as a point mass, whereas the size of mass M is large compared to the distance between the two masses.



In this case, the potential energy is given by

$$V = -Gm \int_M \frac{dM}{r} \ .$$

That is, the total potential energy is the sum of the potential energies due to small elemental masses, dM. The integration must be carried out over the entire mass M, where r is the distance between m and the elemental mass dM being considered.

It turns out that if the mass M, is distributed uniformly over a spherical shell of radius R, then it can be shown, by carrying out the above integral, that:

- The potential when m is *inside the shell* is constant and equal to

  $$V = -G\frac{Mm}{R} \ .$$

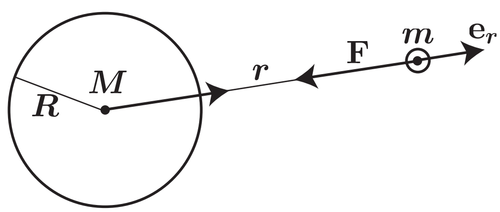
  In this case, we have $\boldsymbol{F} = -\boldsymbol{\nabla} V = \mathbf{0}$

- The potential when m is *outside the shell* is given by

  $$V = -G\frac{Mm}{r} \ ,$$

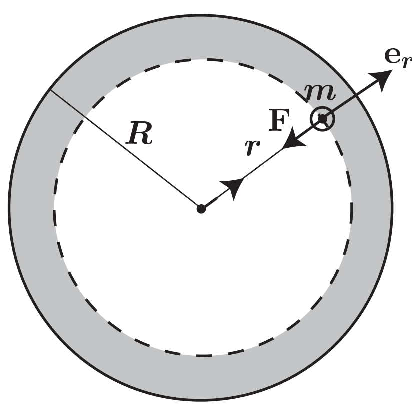
  where r is the distance from m to the center of the shell. In this case, the potential, and, consequently, the force, is identical to that of a point mass M located at the center of the spherical shell.

Therefore, when the mass M is a solid sphere, the gravitational attraction on a mass m, *outside* M, is still given by (1), with r being measured from the sphere center.



If, on the other hand, the mass m is *inside* M, then the attraction force on m due to M, is given by

$$F = G\frac{M'm}{r^2} = G\frac{Mm}{R^2}(\frac{r}{R}) \ .$$



Here, $M' = M(r/R)^3$ is the mass corresponding to a hypothetical sphere of radius r with the same uniform mass density as the original sphere of radius R. In other words, the force of attraction when m is inside M is equal to that of a reduced sphere of radius r instead of R. Thus, we see that the spherical shell outside m, has no effect on the gravitational attraction force on m.

Weight

The gravitational attraction from the Earth to any particle located near the surface of the Earth is called the weight. Thus, the weight, $\mathbf{W}$, of a particle of mass m, is given approximately by

$$\mathbf{W} \approx -G\frac{M_e m}{R_e^2}\boldsymbol{e}_r = -g_0 m \boldsymbol{e}_r = m\mathbf{g}_0 \ .$$

Here, M_e and R are the mass and radius of the Earth, and $\mathbf{g}_0 = -(GM_e/R_e^2)\boldsymbol{e}_r$ is called the *gravitational* acceleration vector.

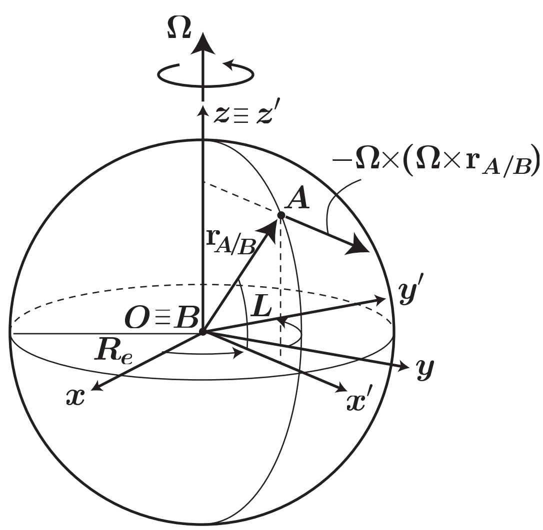
It turns out that the Earth is not quite spherical, and so the weight does not exactly obey the inverse-squared law. The magnitude of the gravitational acceleration, g_0, at the poles and at the equator is slightly different. In addition, the Earth is also rotating. This introduces an inertial centrifugal force which has the effect of reducing the vertical component of the weight.

Note ***Gravity variations due to Earth rotation***

Here, we consider the influence of Earth's rotation on the gravity measured by an observer rotating with the Earth. The starting point will be our general expression for relative motion,

$$\boldsymbol{F} - m\,\boldsymbol{a}_B - 2m\,\boldsymbol{\Omega}\times(\boldsymbol{v}_{A/B})_{x'y'z'} - m\,\dot{\boldsymbol{\Omega}}\times\boldsymbol{r}_{A/B} - m\,\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}_{A/B}) \quad = \quad m\,(\boldsymbol{a}_{A/B})_{x'y'z'} \ . \qquad (2)$$

We consider two reference frames. A fixed frame xyz, and a frame $x'y'z'$ that rotates with the Earth. Both the inertial observer, O, and the rotating observer, B, are situated at the center of the Earth, and are observing a mass m situated at point A on the Earth's surface.

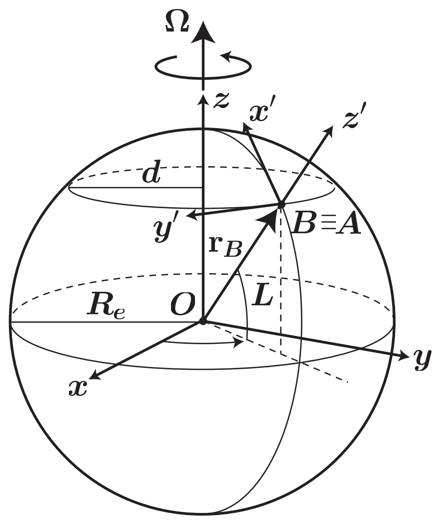


The forces on the mass will be the gravitational force, $m\mathbf{g}_0$, and the reaction force, $\boldsymbol{R}$, which is needed to keep the mass at rest relative to the Earth's surface (if the mass m is placed on a scale, $\boldsymbol{R}$ would be the force that the scale exerts on the mass). Thus, $\boldsymbol{F} = \boldsymbol{R} + m\mathbf{g}_0$. Since the mass m is assumed to be at rest, $\dot{\boldsymbol{\Omega}} = \mathbf{0}$, and, $O \equiv B$, we have,

$$\boldsymbol{R} + m\mathbf{g}_0 - m\,\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_{A/B}) = \mathbf{0}\ , \qquad \text{or,} \qquad \boldsymbol{R} = -m[\mathbf{g}_0 - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_{A/B}] = -m\mathbf{g}$$

Thus, an observer at rest on the surface of the Earth will observe a gravitational acceleration given by $\mathbf{g} = \mathbf{g}_0 - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_{A/B})$. The term $-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_{A/B})$ has a magnitude $\Omega^2 d = \Omega^2 R_e \cos L$, and is directed normal and away from the axis of rotation.

An alternative choice of reference frames which is sometimes more convenient when working with the Earth as a rotating reference frame is illustrated in the figure below.



The fixed xyz axes are the same as before, but now the rotating observer B is situated on the surface of the Earth. A convenient set of rotating axes is that given by *Nort-West-South* directions $x'y'z'$. If we assume that the mass m is located at B, then we have $A \equiv B$, and the above expression (2) reduces to

$$\boldsymbol{R} + m\mathbf{g}_0 - m\,\boldsymbol{a}_B = \mathbf{0}\ , \qquad \text{or,} \qquad \boldsymbol{R} = -m[\mathbf{g}_0 - \boldsymbol{a}_B] = -m\mathbf{g}\ .$$

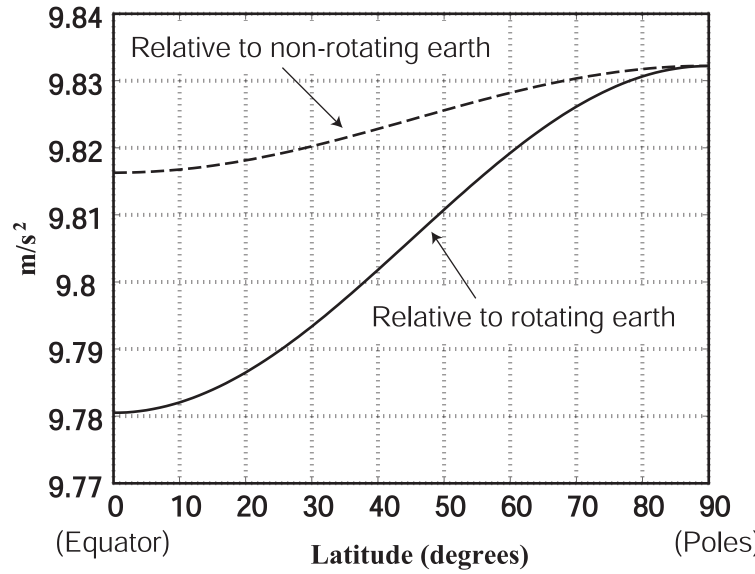
It is straightforward to verify that $\boldsymbol{a}_B = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_B)$, which gives the same expression for $\boldsymbol{R}$, as expected.

If we call $\mathbf{g}$ the *gravity* acceleration vector, which combines the fact that the Earth is not spherical and that it is rotating, the magnitude of g is given by

$$g \approx 9.780327(1 + 0.005279 \sin^2 L + 0.000024 \sin^4 L),$$

where L is the latitude of the point considered and g is given in m/s^2. The coefficient 0.005279 has two components: 0.00344, due to Earth's rotation, and the rest is due to Earth's oblateness (or lack of sphericity).

The higher order term is also due to oblateness. The above expression is known as the international gravity formula and is depicted below.



We note that the gravitational acceleration at the poles is about 0.5% larger than at the equator. Furthermore, the deviations due to the Earth's rotation are about three times larger than the deviations due to the Earth' oblateness.

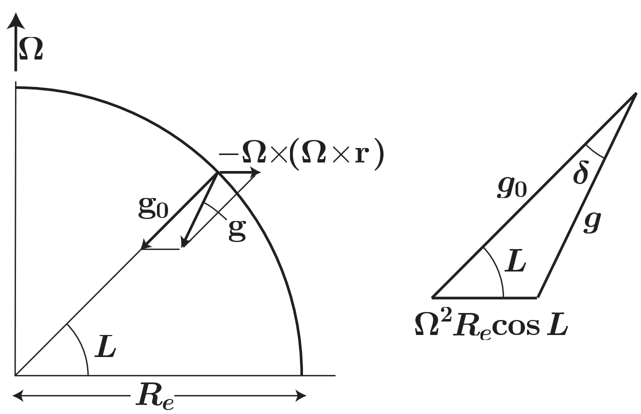
Note ***Angular deviation of*** $\mathbf{g}$

Here, we consider a spherical Earth, and we want to determine the effect of Earth's rotation on the direction $\mathbf{g}$.

In the previous note, we established that an observer rotating with the Earth will observe a gravity vector given by

$$\mathbf{g} = \mathbf{g}_0 - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) \ ,$$

where $\mathbf{g}_0$ is the geocentric gravity, and $\mathbf{g}$ is the modified gravity.



From the triangle formed by g_0, g, and $\Omega^2 R\cos L$, we have $g\sin\delta = \Omega^2 R\cos L\sin L = (\Omega^2 R/2)\sin 2L$. We expect δ to be small, and, therefore, $\sin\delta \approx \delta$, and $g \approx g_0$. Thus,

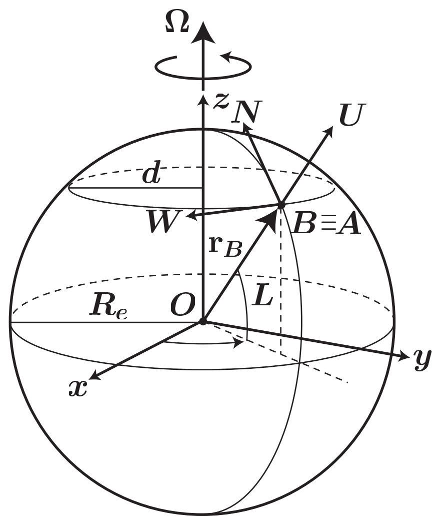
$$\delta \approx \frac{\Omega^2 R_e}{2g}\sin 2L \ ,$$

which is maximum when $L = \pm 45^o$. In this case, we have $\Omega = 7.29(10^{-5})$ rad/s, $R_e = 6370$ km, and $\delta_{max} = 1.7(10^{-3})$ rad $\approx 0.1^o$.

We now consider a couple of three dimensional examples. In the first example, the motion is known, and we are asked to determine the forces required to obtain that motion. In the second example, the motion is unknown and the trajectory needs to be obtained by integrating the equation of motion.

Example ***Aircraft flying at constant velocity***

We now consider an aircraft A, flying with constant velocity $\boldsymbol{v} = v_N\boldsymbol{e}_N + v_W\boldsymbol{e}_W + v_U\boldsymbol{e}_U$ relative to the surface of the Earth. We assume our inertial observer to be at the center of the Earth, and our accelerated observer to be at the aircraft (e.g. $A \equiv B$).



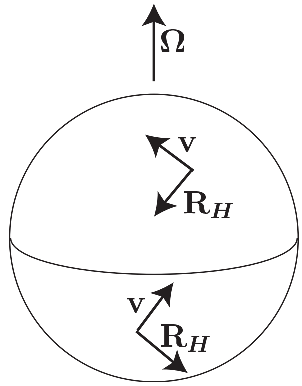
The angular velocity of the Earth, $\boldsymbol{\Omega}$, can be expressed as $\boldsymbol{\Omega} = \Omega\cos L\,\boldsymbol{e}_N + \Omega\sin L\,\boldsymbol{e}_U$. The aerodynamical force, $\boldsymbol{R}$, that an aircraft is required to generate in order to maintain its course is

$$\boldsymbol{R} + m\mathbf{g_0} - m\boldsymbol{a}_B - 2m\boldsymbol{\Omega}\times(\boldsymbol{v})_{NWU} = \mathbf{0} \ .$$

Since $\boldsymbol{a}_B = \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}_B)$, and $\mathbf{g} = \mathbf{g_0} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}_B) \approx -g\boldsymbol{e}_U$, we have,

$$\boldsymbol{R} = -2m\Omega v_W\sin L\,\boldsymbol{e}_N + 2m\Omega(v_N\sin L - v_U\cos L)\,\boldsymbol{e}_W + (mg + 2m\Omega v_W\cos L)\,\boldsymbol{e}_U .$$

For instance, for an aircraft to fly horizontally (i.e. $v_U = 0$), it will require a horizontal force, $\boldsymbol{R}_H = 2m\Omega\sin L(-v_W\,\boldsymbol{e}_N + v_N\,\boldsymbol{e}_W)$. We see that for $L > 0$ (northern hemisphere), this force is always to the "left" of the aircraft, and needs to be generated aerodynamically in order to maintain a straight path. The reverse is true in the southern hemisphere.



For a 200 Ton ($m = 2(10^5)$kg) aircraft at 300 m/s, at a latitude of $L = 42^o$ north, the magnitude of this force is $\boldsymbol{R}_H = 5840$ N $= 1320$ lb.

Note that if this force is not provided, the aircraft will turn to its right.

We also see that for the same aircraft flying east, there is an extra upwards lift of magnitude $2m\Omega v_W \cos L$. For our aircraft, that amounts to 6490 N = 1460 lb, 0.3% of its weight, or about 7 extra passengers.

Example **_Falling object_**

Consider an object being released from a point P situated at a height of 200 m. Calculate the distance between the point of impact and the point at which the plumb line going though P intersects the ground. Neglect air resistance, and assume $L = 45^o$.

We consider the rotating right handed set of axes NWU, and write $\boldsymbol{v} = v_N \boldsymbol{e}_N + v_W \boldsymbol{e}_W + v_U \boldsymbol{e}_U$ and $\boldsymbol{\Omega} = \Omega \cos L\, \boldsymbol{e}_N + \Omega \sin L\, \boldsymbol{e}_U$. The Coriolis acceleration is thus

$$\begin{aligned} \boldsymbol{a}_{cor} &= 2\boldsymbol{\Omega} \times (\boldsymbol{v})_{NWU} = 2 \begin{vmatrix} \boldsymbol{e}_N & \boldsymbol{e}_W & \boldsymbol{e}_U \\ \Omega \cos L & 0 & \Omega \sin L \\ v_N & v_W & v_U \end{vmatrix} \\ &= -2\Omega \sin L v_W\, \boldsymbol{e}_N + 2\Omega(v_N \sin L - v_U \cos L)\, \boldsymbol{e}_W + 2\Omega \cos L v_W\, \boldsymbol{e}_U \end{aligned}$$

Since we expect $v_U \gg v_N, v_W$, we retain only the effect of v_U. Thus, $\boldsymbol{a}_{cor} \approx -2\Omega \cos L v_U\, \boldsymbol{e}_W$. The equation of motion for a falling object will therefore be,

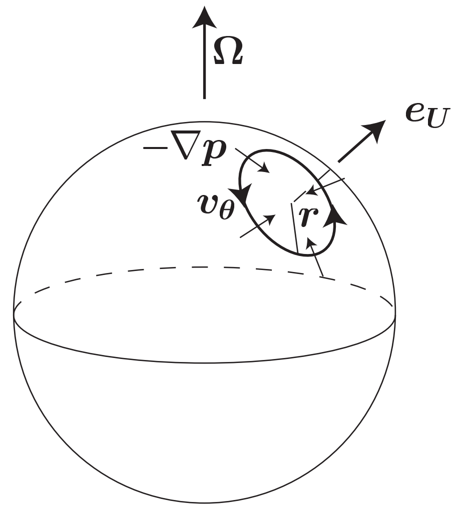
$$-mg\, \boldsymbol{e}_U - m\boldsymbol{a}_{cor} = m(\boldsymbol{a})_{NWU} \ .$$

Here, g includes the centrifugal effects, and $(\boldsymbol{a})_{NWU}$ is the acceleration experienced by a non-inertial observer on the Earth's surface.

In the $\boldsymbol{e}_U$ direction, we have $a_U = -g$, $v_U = -gt$, and $x_U = 200 - (gt^2)/2$. The time required for the object to reach the ground will be obtained for $x_U = 0$, which gives $t = 6.4$ s. In the $\boldsymbol{e}_W$ direction, we have $a_W = 2\Omega \cos L v_U = -2\Omega g \cos L t$, $v_W = -\Omega g \cos L t^2$, and $x_W = -\Omega g \cos L (t^3/3)$. For $t = 6.4$ s, this gives $x_W = -0.044$ m ≈ -4 cm.

Example (Adapted from MMS) **(Optional) Cyclonic Air Motion**

Suppose that there is an area of low pressure in the Northern Hemisphere, so that the pressure force per unit mass on an air element is $-\boldsymbol{\nabla} p/\rho$ and is radially inwards. One would think that the air should rush in radially under this force to "fill in the hole".



Instead, the wind may be such that air moves in circular paths around the depression. The radial acceleration is then, $(a_r)_{x'y'z'} = -r\dot{\theta}^2 = -v_\theta^2/r$. The real force acting radially per unit mass is, as noted, $-(1/\rho)dp/dr$, and, in addition, we would have to include the inertial forces due to Earth's rotation, namely, a Coriolis force $-2\boldsymbol{\Omega} \times (\boldsymbol{v})_{x'y'z'}$ per unit mass. The combined effect of gravity and Earth's centrifugal force acts in the direction of the local vertical. Therefore, the equation of motion in the radial direction is

$$-\frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{dp}{dr} + 2\Omega_U v_\theta \ .$$

Here, $\Omega_U = \boldsymbol{\Omega} \cdot \boldsymbol{e}_U$ is the component of the angular velocity in the vertical direction. If we compare the magnitude of the acceleration term with that due to Coriolis' effect,

$$\frac{2\Omega_U v_\theta}{v_\theta^2/r} = 2\Omega_U \frac{r}{v_\theta} \ ,$$

we see that for a given Ω_U, Coriolis' effect becomes important for large values of r/v_θ. Therefore, we consider two limits:

- Large values of r/v_θ. This leads to the so called *Geostrophic Winds*. In this case, the acceleration term is small and the approximate governing equation becomes,

  $$0 = -\frac{1}{\rho}\frac{dp}{dr} + 2\Omega_U v_\theta \ .$$

  Consider for instance $v_\theta = 10$ m/s and r in the range of 100 to 400 km. At a latitude of 42^o, we have $\Omega_U \approx 4.9 \times 10^{-5}$ rad/s, and $dp/dr = -2\rho\Omega_U v_\theta \approx 0.0012$ N/m^3, or 1.2 mb per 100 km, a moderate pressure gradient. These winds are responsible for most regional weather patterns; they circulate counterclockwise in the Northern Hemisphere (and clockwise in the Southern Hemisphere).

- Small values of r/v_θ. This limit includes the *tornadoes*. Pressure defects of the order of 0.1 atm $\approx 0.1 \times 10^5$ b can occur in a tornado over scales of the order of 10 m. In this case, Coriolis' effects become unimportant and the governing equation reduces to

$$-\frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{dp}{dr} \ .$$

Typical values for the velocity are,

$$v_\theta = \sqrt{\frac{r}{\rho}\frac{dp}{dr}} \approx 100\text{m/s} \ .$$

References

[1] M. Martinez-Sanchez, *Unified Engineering Notes*, Course 95-96.