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16.07 Dynamics
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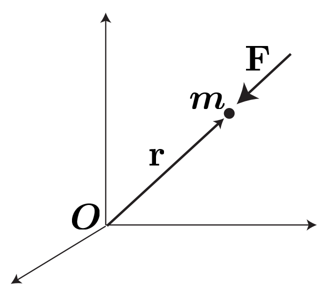
Lecture D28 - Central Force Motion: Kepler's Laws

When the only force acting on a particle is always directed towards a fixed point, the motion is called *central force motion*. This type of motion is particularly relevant when studying the orbital movement of planets and satellites. The laws which govern this motion were first postulated by Kepler and deduced from observation. In this lecture, we will see that these laws are a consequence of Newton's second law. An understanding of central force motion is necessary for the design of satellites and space vehicles.

Kepler's Problem

We consider the motion of a particle of mass m, in an inertial reference frame, under the influence of a force, $\boldsymbol{F}$, directed towards the origin.



We will be particularly interested in the case when the force is inversely proportional to the square of the distance between the particle and the origin, such as the gravitational force. In this case,

$$\boldsymbol{F} = -\frac{\mu}{r^2} m \boldsymbol{e}_r ,$$

where μ is the gravitational parameter, r is the modulus of the position vector, $\boldsymbol{r}$, and $\boldsymbol{e}_r = \boldsymbol{r}/r$.
It can be shown that, in general, Kepler's problem is equivalent to the two-body problem, in which two masses, M and m, move solely due to the influence of their mutual gravitational attraction. This equivalence is obvious when $M >> m$, since, in this case, the center of mass of the system can be taken to be at M. However, even in the more general case when the two masses are of similar size, the problem can be reduced to a Kepler problem (see note below).

Although most problems in celestial mechanics involve more than two bodies, many problems of practical interest can be accurately solved by just looking at two bodies at a time. When more than two bodies are involved, the problem is considerably more complicated, and, in this case, no general solutions are known. This problem was studied by Kepler (1571-1630) who lived before Newton was born. His interest was in describing the motion of planets around the sun. He postulated the following laws:

1.- The orbits of the planets are ellipses with the Sun at one focus

2.- The line joining a planet to the Sun sweeps out equal areas in equal intervals of time

3.- The square of the period of a planet is proportional to the cube of the major axis of its elliptical orbit

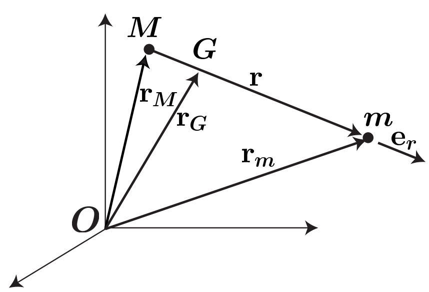
In this lecture, we will start from Newton's laws and verify that the above three laws can indeed be derived from Newtonian mechanics.

Note — ***Equivalence between the two-body problem and Kepler's problem***

Here we consider the problem of two isolated bodies of masses M and m which interact though gravitational attraction. Let $\boldsymbol{r}_M$ and $\boldsymbol{r}_m$ denote the position vectors of the two bodies relative to a fixed origin O. Thus, the position of the center of gravity, G, of the two bodies will be

$$\boldsymbol{r}_G = \frac{M\boldsymbol{r}_M + m\boldsymbol{r}_m}{M+m} \ .$$

Since the two bodies are isolated, we will have, from momentum conservation, that $\dot{\boldsymbol{r}}_G =$*constant*, and $\ddot{\boldsymbol{r}}_G = \boldsymbol{0}$. Therefore, the position of the center of gravity, at all times, can be found trivially from the initial conditions.



If the position vector of m as observed by M, $\boldsymbol{r} = \boldsymbol{r}_m - \boldsymbol{r}_M$, is known, then the position vectors of M and m could be computed as

$$\boldsymbol{r}_M = \boldsymbol{r}_G - \frac{m}{M+m}\boldsymbol{r}, \qquad \boldsymbol{r}_m = \boldsymbol{r}_G + \frac{M}{M+m}\boldsymbol{r} \ .$$

Therefore, the problem of determining $\boldsymbol{r}_M$ and $\boldsymbol{r}_m$ is equivalent to that of determining $\boldsymbol{r}$. In order to derive an equation for $\boldsymbol{r}$, we first consider the equations of motion for m and M independently. Thus, we have

$$M\ddot{\boldsymbol{r}}_M = G\frac{Mm}{r^2}\boldsymbol{e}_r, \qquad m\ddot{\boldsymbol{r}}_m = -G\frac{Mm}{r^2}\boldsymbol{e}_r \ ,$$

where $r = |\boldsymbol{r}|$, $\boldsymbol{e}_r = \boldsymbol{r}/r$, and G is the gravitational constant. Subtracting these two expressions, we obtain,

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{r}}_m - \ddot{\boldsymbol{r}}_M = -G\frac{M+m}{r^2}\,\boldsymbol{e}_r \ ,$$

or,

$$\frac{Mm}{M+m}\ddot{\boldsymbol{r}} = -G\frac{Mm}{r^2}\,\boldsymbol{e}_r \ .$$

The above expression shows that the motion of m relative to M is in fact a Kepler problem in which the force is given by $-GMm\boldsymbol{e}_r/r^2$ (this is indeed the real force), but the mass of the orbiting body (m in this case), has been replaced by the *reduced mass*, $Mm/(M+m)$. Note that when $M >> m$, the reduced mass becomes m. However, the above expression is general and applies to general masses M and m.
Alternatively, the above expression can be written as

$$m\ddot{\boldsymbol{r}} = -G\frac{(M+m)m}{r^2}\,\boldsymbol{e}_r \ ,$$

which is again a Kepler problem for an orbiting body of mass M, in which the gravitational parameter μ is given by $\mu = G(M+m)$.

Equations of Motion

The equation of motion ($\boldsymbol{F} = m\boldsymbol{a}$), is

$$-\frac{\mu m}{r^2}\boldsymbol{e}_r = m\ddot{\boldsymbol{r}}.$$

Since the only force in the system is directed towards point O, the angular momentum of m with respect to the origin will be constant. Therefore, the position and velocity vectors, $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$, will be in a plane orthogonal to the angular momentum vector, and, as a consequence, the motion will be planar. Using cylindrical coordinates, with $\boldsymbol{e}_z$ being parallel to the angular momentum vector, we have,

$$-\frac{\mu}{r^2}\boldsymbol{e}_r = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta.$$

Now, we consider the radial and circumferential components of this equation separately.

Circumferential component

We have,

$$0 = r\ddot{\theta} + 2\dot{r}\dot{\theta} \ .$$

Using the following identity,

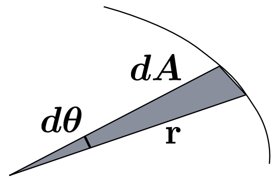
$$\frac{1}{r}\left(\frac{d}{dt}(r^2\dot{\theta})\right) = r\ddot{\theta} + 2\dot{r}\dot{\theta},$$

the above equation implies that

$$r^2\dot{\theta} = h \equiv \text{constant.} \tag{1}$$

We note that the constant of integration, h, that will be determined by the initial conditions, is precisely the magnitude of the specific angular momentum vector, i.e. $h = |\boldsymbol{r} \times \boldsymbol{v}|$.

In a time dt, the area, dA, swept by $\boldsymbol{r}$ will be $dA = r\, rd\theta/2$.



Therefore,

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{h}{2},$$

which proves *Kepler's second law.*

Radial component

The radial component of the equation of motion reads,

$$-\frac{\mu}{r^2} = \ddot{r} - r\dot{\theta}^2. \tag{2}$$

Since $-r^2\frac{d}{dt}\left(\frac{1}{r}\right) = \dot{r}$, and $r^2 = h/\dot{\theta}$, from equation 1, we can write

$$\dot{r} = -\frac{h}{\dot{\theta}}\frac{d}{dt}\left(\frac{1}{r}\right) = -h\frac{d}{d\theta}\left(\frac{1}{r}\right).$$

Differentiating with respect to time,

$$\ddot{r} = -h\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\dot{\theta} = -\frac{h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right).$$

Inserting this expression into equation 2, and using equation 1, we obtain the following differential equation for $1/r$ as a function of θ.

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{\mu}{h^2}.$$

This is a linear second order ordinary differential equation which has a general solution of the form,

$$\frac{1}{r} = \frac{\mu}{h^2}(1 + e\cos(\theta + \psi)) \quad ,$$

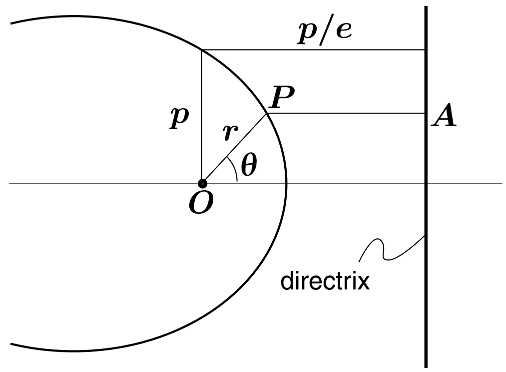
where e and ψ are two constants of integration. If we *choose θ to be zero when r is minimum*, then e will be positive, and $\psi = 0$. The equation describing the trajectory will be

$$r = \frac{h^2/\mu}{1 + e\cos\theta} \ . \tag{3}$$

We shall see below that this is the equation of a conic section in polar coordinates.

Conic Sections

Conic sections are planar curves that are defined as follows: given a line, or directrix, and a point, or focus O, a conic section is the locus of points, P, such that the ratio of the distance between the point and the focus, PO, to the distance between the point and the directrix, PA, is a constant e. That is, $e = PO/PA$.



Since, $PO = r$, and $PA = p/e - r\cos\theta$, we have

$$r = \frac{p}{1 + e\cos\theta} \ . \tag{4}$$

Here, p is the so called parameter of the conic and is equal to r when $\theta = \pm 90^o$. The constant $e \geq 0$ is called the eccentricity, and, depending on its value, the conic surface will be either an open or closed curve. In particular, we have that when

$e = 0$	the curve is a circle
$e < 1$	the curve is an ellipse
$e = 1$	the curve is a parabola
$e > 1$	the curve is a hyperbola.

Comparing equations 4 and 3, we see that the trajectory of a mass under the influence of a central force will be a conic curve with parameter $p = h^2/\mu$. When $e < 1$, the trajectory is an ellipse, thus proving *Kepler's first law.*

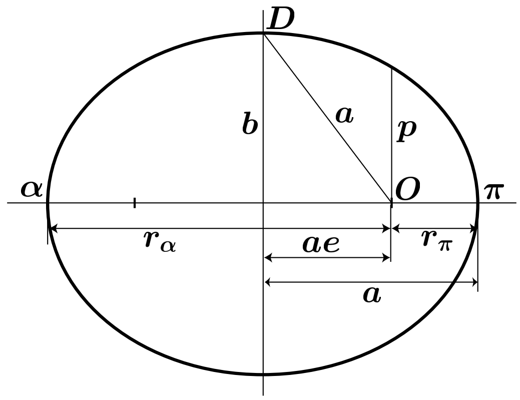
The point in the trajectory which is closest to the focus is called the *periapsis* and is denoted by π. For elliptical orbits, the point in the trajectory which is farthest away from the focus is called the *apoapsis* and is denoted by α. When considering orbits around the earth, these points are called the *perigee* and *apogee*, whereas for orbits around the sun, these points are called the *perihelion* and *aphelion*, respectively.

Elliptical Trajectories

If a is the semi-major axis of the ellipse, then $2a = r_\pi + r_\alpha$, using equation 4 to evaluate r_π and r_α, we obtain $a = p/(1 - e^2)$. Thus,

$$r_\pi = \frac{p}{1+e} = a(1-e), \qquad r_\alpha = \frac{p}{1-e} = a(1+e).$$

Also, the distance between O and the center of the ellipse will be $a - r_\pi = a\,e$.



The distance between point D and the directrix will be equal to DO/e, which in turn will be equal to the sum of the distance between the focus and the center of the ellipse, plus the distance between the focus and the directrix. That is, $DO/e = ae + p/e$. Therefore, $DO = a\,e^2 + p = a$. Hence, using Pythagoras' theorem, $b^2 + (a\,e)^2 = a^2$, the semi-minor axis of the ellipse will be $b = a\sqrt{1-e^2}$.

The area of the ellipse is given by $A = \pi ab$. Also, since $dA/dt = h/2$ is a constant, we have $A = hT/2$, where T is the period of the orbit. Equating these two expressions and expressing h in terms of the semi-major axis as $h^2 = \mu p = \mu a(1-e^2)$, we have

$$\mu = \left(\frac{2\pi}{T}\right)^2 a^3, \tag{5}$$

which proves *Kepler's third law.*

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition
3/13 (except energy analysis)