

## Lecture D11 - Relative Motion using Translating Axes

In the previous lectures we have described particle motion as it would be seen by an observer standing still at a fixed origin. This type of motion is called *absolute* motion. In many situations of practical interest, we find ourselves forced to describe the motion of bodies while we are simultaneously moving with respect to a more basic reference. There are many examples where such situations occur. The absolute motion of a passenger inside an aircraft is best described if we first consider the motion of the passenger relative to the aircraft, and then the motion of the aircraft relative to the ground. If we try to track the motion of aircraft in the airspace using satellites, it makes sense to first consider the motion of the aircraft relative to the satellite and then combine this motion with the motion of the satellite relative to the earth's surface. In this lecture we will introduce the ideas of *relative* motion analysis.

### Types of observers

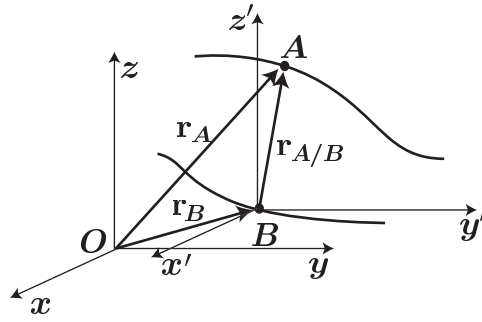
For the purpose of studying relative motion, we will consider three different types of observers (or reference frames) depending on their motion with respect to a fixed frame:

- observers who do not accelerate or rotate, i.e. those who at most have constant velocity.
- observers who accelerate but do not rotate
- observers who accelerate and rotate

In this lecture we will consider the relative motion involving observers of the first two types, and defer for the next lecture the study of relative motion involving rotating frames.

### Relative motion using translating axes

Consider two particles,  $A$  and  $B$ , which may have different curvilinear motions. We consider a fixed reference frame  $xyz$  with origin  $O$  and with unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$ , as before, and call the motion relative to this frame absolute. In addition, we consider another *translating* reference frame attached to particle  $B$ ,  $x'y'z'$ , with unit vectors  $\mathbf{i}'$ ,  $\mathbf{j}'$  and  $\mathbf{k}'$ . Translating means that the angles between the axes  $xyz$  and  $x'y'z'$  do not change during the motion.



In the figure, we have chosen, for convenience, the axes  $xyz$  to be parallel to the axes  $x'y'z'$ , but it should be clear that one could have non-parallel translating axes.

The position vector  $\mathbf{r}_{A/B}$  defines the position of  $A$  with respect to point  $B$  in the reference frame  $x'y'z'$ . The subscript notation “ $A/B$ ” means “ $A$  relative to  $B$ ”. The positions of  $A$  and  $B$  relative to the absolute frame are given by the vectors  $\mathbf{r}_A$  and  $\mathbf{r}_B$ , respectively. Thus, we have

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B} .$$

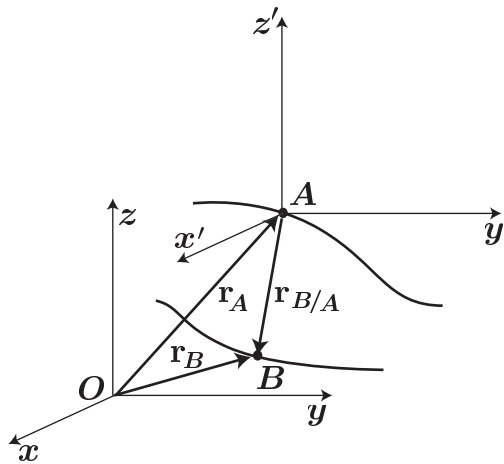
If we derive this expression with respect to time, we obtain

$$\dot{\mathbf{r}}_A = \dot{\mathbf{r}}_B + \dot{\mathbf{r}}_{A/B} \quad \text{or} \quad \mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B} ,$$

which relates the absolute velocities  $\mathbf{v}_A$  and  $\mathbf{v}_B$  to the relative velocity of  $A$  as observed by  $B$ . Differentiating again, we obtain an analogous expression for the accelerations,

$$\ddot{\mathbf{r}}_A = \ddot{\mathbf{r}}_B + \ddot{\mathbf{r}}_{A/B} \quad \text{or} \quad \mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B} .$$

If we reverse the roles of  $A$  and  $B$  and attach the reference frame  $x'y'z'$  to  $A$ , then we can observe  $B$  from  $A$ .



The same arguments as before will give us,

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}, \quad \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}, \quad \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} .$$

Comparing these expressions with those above, we see that

$$\mathbf{r}_{B/A} = -\mathbf{r}_{A/B}, \quad \mathbf{v}_{B/A} = -\mathbf{v}_{A/B}, \quad \mathbf{a}_{B/A} = -\mathbf{a}_{A/B},$$

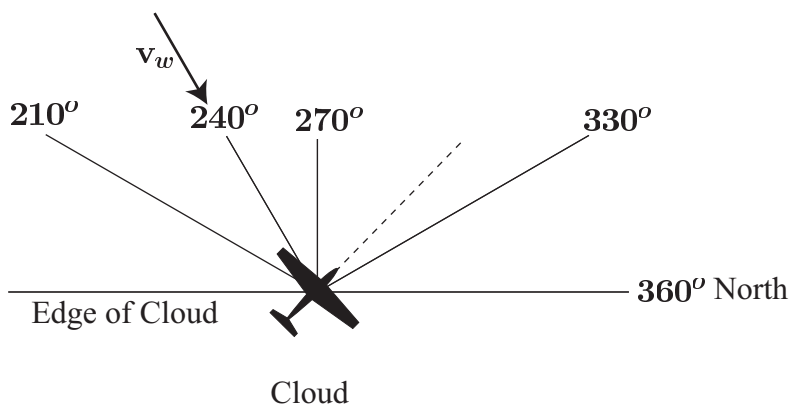
as expected.

One important observation is that, whenever the moving system, say  $A$ , has a constant velocity relative to the fixed system,  $O$ , then the acceleration seen by the two observers is the same, i.e., if  $\mathbf{a}_A = \mathbf{0}$ , then  $\mathbf{a}_B = \mathbf{a}_{B/A}$ . We shall see that this broadens the application of Newton's second law to systems which have a constant absolute velocity.

**Example**

***Glider in cross wind***

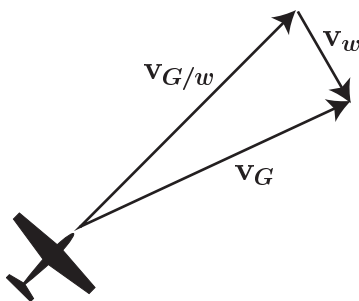
Consider a glider flying at the edge of a cloud. The glider is flying horizontally, and the cloud, which is below the glider, is stationary with respect to the ground. At the altitude of the glider flight, there is a wind velocity,  $\mathbf{v}_w$ , of magnitude 52 knots, as shown in the sketch.



The glider, on the other hand, is flying at a speed  $\mathbf{v}_{G/w}$  relative to the local wind. In order to determine the speed of the glider with respect to the ground, we consider a reference frame moving with the wind speed and write

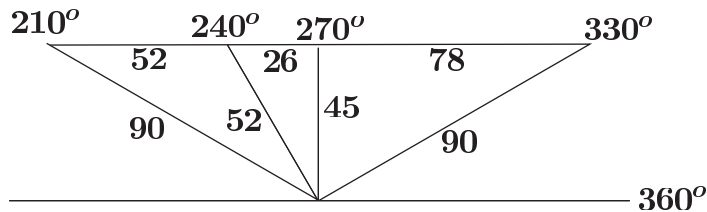
$$\mathbf{v}_G = \mathbf{v}_w + \mathbf{w}_{G/w},$$

or graphically,



By looking at the scaled diagram below, we can verify the following situations. If the glider flies with a heading of  $330^\circ$  and a speed (relative to the wind) of 90 knots, it will stay at the edge of the cloud with a

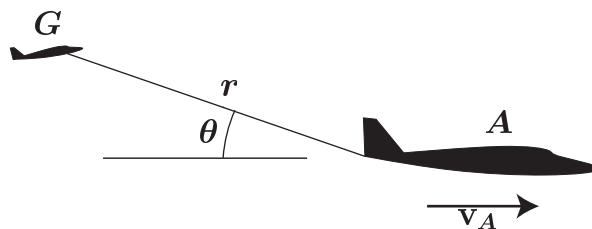
ground speed of  $104 = (26 + 78)$  knots north. If, on the other hand, the glider flies with a heading of  $210^\circ$  at 90 knots, the glider will stay at the edge of the cloud with a ground speed of 52 knots south. If the glider flies into the wind at  $240^\circ$ , with a speed equal to 52 knots, it will remain stationary with respect to the ground. Finally, the lowest speed at which the glider can fly to stay at the edge of the cloud is 45 knots and this will be possible for a heading of  $270^\circ$ . The ground speed in this case will be 24 knots.



**Example**

**Aircraft towing glider (Merriam)**

Airplane  $A$  is flying horizontally with a constant speed of 200km/h, and is towing a glider  $B$ . The glider is gaining altitude. The tow cable has a length  $r = 60\text{m}$ , and  $\theta$  is increasing at a constant rate of 5 degrees per second. We want to determine the magnitude of the velocity,  $\mathbf{v}_G$ , and the acceleration,  $\mathbf{a}_G$ , of the glider for the instant when  $\theta = 15^\circ$ .



The velocity of the glider will be  $\mathbf{v}_G = \mathbf{v}_A + \mathbf{v}_{G/A}$ . The velocity of the glider relative to  $A$  is best expressed using a polar coordinate system. Thus, we write,

$$\mathbf{v}_{G/A} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta .$$

Since the length of the cable is constant,  $v_r = \dot{r} = 0$ , and  $\dot{\theta} = 5(2\pi/360) = 0.0874$  rad/sec, which gives  $v_\theta = r\dot{\theta} = 5.236$  m/s. We can now transform the velocity vector to cartesian coordinates to obtain,

$$\mathbf{v}_{G/A} = 5.236\mathbf{e}_\theta = 5.236 \sin 15^\circ\mathbf{i} + 5.236 \cos 15^\circ\mathbf{j} = 1.3552\mathbf{i} + 5.0575\mathbf{j} \text{ m/s} ,$$

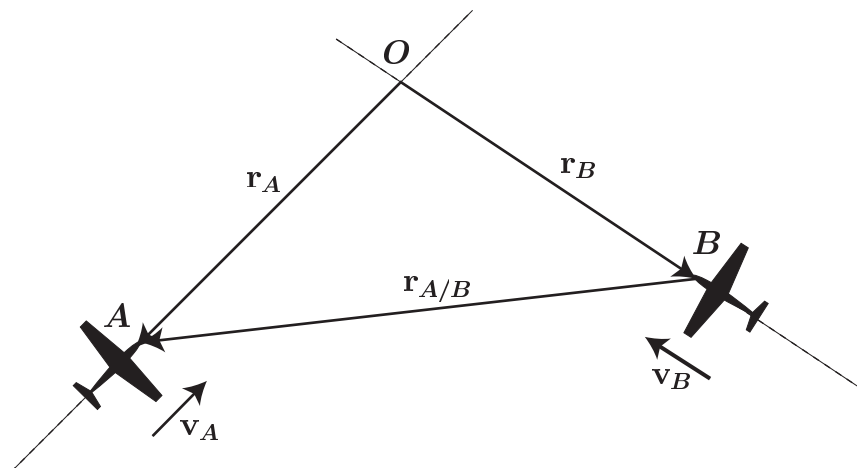
or,

$$\mathbf{v}_G = ((200/3.6) + 1.3552)\mathbf{i} + 5.0575\mathbf{j} \text{ m/s}, \quad v_G = 57.135 \text{ m/s} = 205.7 \text{ km/h} .$$

For the acceleration  $\mathbf{a}_G = \mathbf{a}_{G/A}$ , since  $\mathbf{a}_A = \mathbf{0}$ . We also know that  $\ddot{\theta} = 0$ , and, obviously,  $\ddot{r} = 0$ , thus,

$$\mathbf{a}_{G/A} = -r\dot{\theta}^2\mathbf{e}_r = -4.57\mathbf{e}_r \text{ m/s}^2, \quad a_{G/S} = 4.57 \text{ m/s}^2 .$$

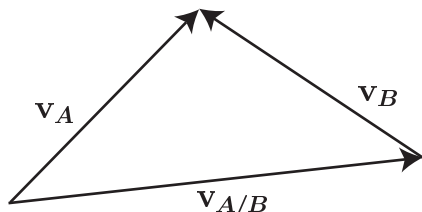
Consider two aircraft,  $A$  and  $B$ , flying horizontally at the same level with constant velocities  $\mathbf{v}_A$  and  $\mathbf{v}_B$ . Aircraft  $B$  is capable of determining by measurement the relative position and velocity of  $A$  relative to  $B$ , i.e.,  $\mathbf{r}_{A/B}$  and  $\mathbf{v}_{A/B}$ . We want to know whether it is possible to determine from this measurement whether the two aircraft are on a collision course.



Take the origin  $O$  at the hypothetical point of collision where the two trajectories intersect. If collision is to occur, the time taken for both aircraft to reach  $O$  will be the same. Therefore,

$$\frac{r_A}{v_A} = \frac{r_B}{v_B}.$$

Now consider the velocity triangle formed by  $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$ .



Clearly,  $\mathbf{v}_A$  is parallel to  $\mathbf{r}_A$ , and  $\mathbf{v}_B$  is parallel to  $\mathbf{r}_B$ . In addition, if collision is to occur,  $\mathbf{v}_{A/B}$  will be parallel to  $\mathbf{r}_{A/B}$ , since the position and velocity triangles will, in this case, be similar. We see, therefore, that the condition for the two planes to collide is that the relative velocity between them is parallel to the relative position vector. If aircraft  $B$  uses a polar coordinate system (common for radar measurements), then the relative position of  $A$  with respect to  $B$  will be of the form  $\mathbf{r}_{A/B} = r\mathbf{e}_r$ . Therefore, the velocity also has to be of that form, which implies that  $v_\theta = r\dot{\theta} = 0$ , or  $\theta = \text{constant}$ . That is, the angle under which  $B$  sees  $A$  (sometimes called the bearing angle) should not change.

**ADDITIONAL READING**

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

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