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Lecture D9 - Linear Impulse and Momentum

In this lecture we will consider the equations that result from integrating Newton's second law, $\boldsymbol{F} = m\boldsymbol{a}$, in time. This will lead to the principle of linear impulse and momentum. This principle is very useful when solving problems in which we are interested in determining the global effect of a force acting on a particle over a time interval.

Linear Momentum

We consider the curvilinear motion of a particle of mass, m, under the influence of a force $\boldsymbol{F}$. Assuming that the mass does not change, we have from Newton's second law,

$$\boldsymbol{F} = m\boldsymbol{a} = m\frac{d\boldsymbol{v}}{dt} = \frac{d}{dt}(m\boldsymbol{v}) \ .$$

The case where the mass of the particle changes with time (e.g. a rocket) will be considered later on in this course. The *linear momentum* vector, $\boldsymbol{L}$, is defined as

$$\boldsymbol{L} = m\boldsymbol{v} \ .$$

Thus, an alternative form of Newton's second law is

$$\boldsymbol{F} = \dot{\boldsymbol{L}} \ , \tag{1}$$

which states that the total force acting on a particle is equal to the time rate of change of its linear momentum.

Principle of Linear Impulse and Momentum

Imagine now that the force considered acts on the particle between time t_1 and time t_2. Equation (1) can then be integrated in time to obtain

$$\int_{t_1}^{t_2} \boldsymbol{F} \ dt = \int_{t_1}^{t_2} \dot{\boldsymbol{L}} \ dt = \boldsymbol{L}_2 - \boldsymbol{L}_1 = \Delta \boldsymbol{L} \ . \tag{2}$$

Here, $\boldsymbol{L}_1 = \boldsymbol{L}(t_1)$ and $\boldsymbol{L}_2 = \boldsymbol{L}(t_2)$. The term

$$\boldsymbol{I} = \int_{t_1}^{t_2} \boldsymbol{F} \ dt \ ,$$

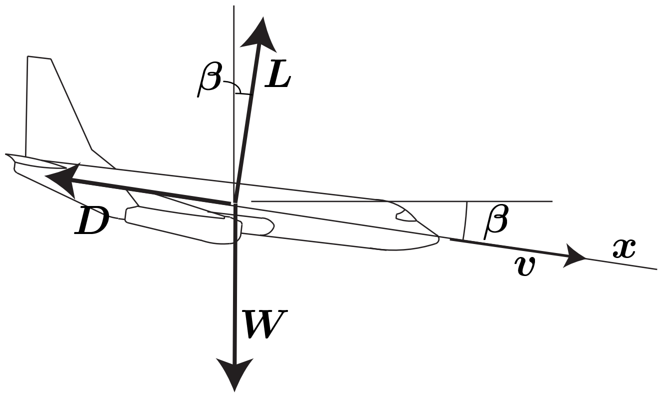
is called the *linear impulse*. Thus, the linear impulse on a particle is equal to the linear momentum change.

Note **Units of Impulse and Momentum**

It is obvious that linear impulse and momentum have the same units. In the SI system they are N · s or kg · m/s, whereas in the English system they are lb · s, or slug · ft/s.

Example (MK) **Average Drag Force**

The pilot of a 90,000-lb airplane which is originally flying horizontally at a speed of 400 mph cuts off all engine power and enters a glide path as shown where $\beta = 5^o$. After 120 s, the airspeed of the plane is 360 mph. We want to calculate the magnitude of the time-averaged drag force.



Aligning the x-axis with the flight path, we can write the x component of equation (2) as follows

$$\int_0^{120} (W \sin\beta - D)\, dt = L_x(120) - L_x(0) \ .$$

The time-averaged value of the drag force, $\bar{D}$, is

$$\bar{D} = \frac{1}{120} \int_0^{120} D \, dt \ .$$

Therefore,

$$(W \sin\beta - \bar{D})120 = m(v_x(120) - v_x(0)) \ .$$

Substituting and applying the appropriate unit conversion factors we obtain,

$$(90,000 \sin 5^o - \bar{D})120 = \frac{90,000}{32.2}(360 - 400)\frac{5280}{3600} \quad \rightarrow \quad \bar{D} = 9,210 \text{ lb} \ .$$

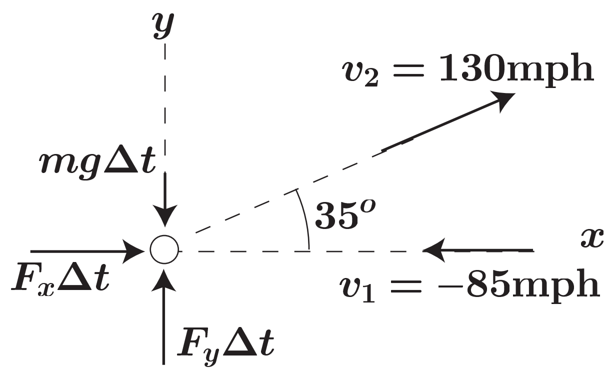
Impulsive Forces

We typically think of impulsive forces as being forces of very large magnitude that act over a very small interval of time, but cause a significant change in the momentum. Examples of impulsive forces are those generated when a ball is hit by a tennis racquet or a baseball bat, or when a steel ball bounces on a steel plate. The table below shows typical time intervals over which some of these impulses occur.

	Time interval Δt [s]
Racquet hitting a tennis ball	0.005 – 0.05
Bat hitting a baseball	0.01–0.02
Golf club hitting a golf ball	0.001
Pile Driver	0.01 – 0.02
Shotgun	0.001
Steel ball bouncing on steel plate	0.0002

Example (MK) ***Baseball Bat***

A baseball is travelling with a horizontal velocity of 85 mph just before impact with the bat. Just after the impact, the velocity of the $5\frac{1}{8}$ oz. ball is 130 mph, at an angle of 35^o above the horizontal. We want to determine the horizontal and vertical components of the average force exerted by the bat on the baseball during the 0.02 s impact.



If we consider the time interval between the instant the baseball hits the bat, t_1, and the instant after it leaves the bat, t_2, the forces responsible for changing the baseball's momentum are gravity and the contact forces exerted by the bat. We can use the x and y components of equation (2) to determine the average force of contact. First, consider the x component,

$$mv_{x1} + \int_{t_1}^{t_2} F_x \, dt = mv_{x2} \ .$$

Inserting numbers and using appropriate conversion factors, we obtain

$$-\frac{5.125/16}{32.2}(85\frac{5280}{3600}) + \bar{F}_x(0.02) = \frac{5.125/16}{32.2}(130\frac{5280}{3600}\cos 35^o) \quad \rightarrow \quad \bar{F}_x = 136.7 \text{ lb} \ .$$

For the y direction, we have

$$mv_{y1} + \int_{t_1}^{t_2} F_y \, dt = mv_{y2} \ ,$$

which gives,

$$0 + \bar{F}_y(0.02) - \frac{5.125}{16}(0.02) = \frac{5.125/16}{32.2}(130\frac{5280}{3600}\sin 35^o) \quad \rightarrow \quad \bar{F}_y = 54.7 \text{ lb} \ .$$

We note that $mg\Delta t = 0.0064$ lb-s, whereas $\bar{F}_y(\Delta t) = 1.094$ lb - s. Thus, $mg\Delta t$ is 0.59% of the total impulse. We could have safely neglected it. In this case we would have obtained $\bar{F}_y = 54.4$ lb.

Conservation of Linear Momentum

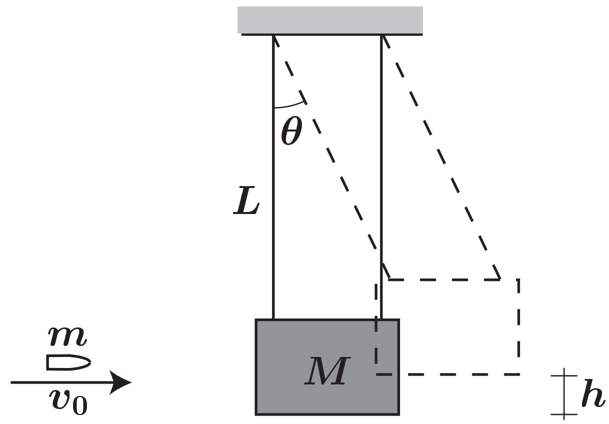
We see from equation (1) that if the resultant force on a particle is zero during an interval of time, then its linear momentum $\boldsymbol{L}$ must remain constant. Since equation (1) is a vector quantity, we can have situations in which only some components of the resultant force are zero. For instance, in Cartesian coordinates, if the resultant force has a non-zero component in the y direction only, then the x and z components of the linear momentum will be conserved since the force components in x and z are zero.

Consider now two particles, m_a and m_b, which interact during an interval of time. Assume that interaction forces between them are the only unbalanced forces on the particles. Let $\boldsymbol{F}$ be the interaction force that particle m_b exerts on particle m_a. Then, according to Newton's third law, the interaction force that particle m_a exerts on particle m_b will be $-\boldsymbol{F}$. Using expression (2), we will have that $\Delta \boldsymbol{L}_a = -\Delta \boldsymbol{L}_b$, or $\Delta \boldsymbol{L} = \Delta \boldsymbol{L}_a + \Delta \boldsymbol{L}_b = \boldsymbol{0}$. That is, the changes of momentum of particles m_a and m_b are equal in magnitude and opposite in sign, and the total momentum change equals zero. Recall that this is true if the only unbalanced forces on the particles are the interaction forces. The more general situation in which external forces can be present will be considered in future lectures.

We note that the above argument is also valid in a componentwise sense. That is, when two particles interact and there are no external unbalanced forces along a given direction, then the total momentum change along that direction must be zero.

Example — ***Ballistic Pendulum***

The ballistic pendulum is used to measure the velocity of a projectile by observing the maximum angle $\theta_{\max}$ to which the box of sand with the embedded projectile swings. Find an expression that relates the initial velocity of a projectile v_0 of mass m to the maximum angle $\theta_{\max}$ reached by the pendulum. The mass of the sand box is M and the length of the pendulum is L.



We consider the equation of conservation of linear momentum along the horizontal direction. The initial momentum of the projectile is mv_0. Since the sand box is initially at rest its momentum is zero. Just after the projectile penetrates into the box, the velocity of the sand box and the projectile are the same. Therefore, if v_1 is the velocity of the sand box (with the embedded projectile) just after impact, we have from conservation of momentum,

$$mv_0 = (M + m)v_1 \ .$$

After impact, the problem reduces to that of a simple pendulum. The only force doing any work is gravity and therefore we can apply the principle of conservation of work and energy. At the point when θ is maximum, the velocity will be zero. From energy conservation we finally obtain,

$$\frac{1}{2}(M + m)v_1^2 = (M + m)gh_{\max} \ ,$$

and since $h = L(1 - \cos\theta)$, we have

$$\theta_{\max} = \cos^{-1}\left(1 - \left(\frac{m}{M + m}\right)^2 \frac{v_0^2}{2Lg}\right) \ .$$

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

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