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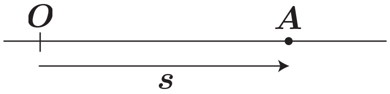
Dynamics 16.07
Fall 2004
Version 1.1

REVIEW - Rectilinear Motion

We start by considering the simple motion of a particle along a straight line. The position of particle A at any instant can be specified by the coordinate s with origin at some fixed point O.



The instantaneous velocity is

$$v = \frac{ds}{dt} = \dot{v} \ . \tag{1}$$

We will be using the "dot" notation, to indicate time derivative, e.g. $(\dot{\ }) \equiv d/dt$. Here, a positive v means that the particle is moving in the direction of increasing s, whereas a negative v, indicates that the particle is moving in the opposite direction. The acceleration is

$$a = \frac{dv}{dt} = \dot{v} = \frac{d^2 s}{dt^2} = \ddot{s} \ . \tag{2}$$

The above expression allows us to calculate the speed and the acceleration if s and/or v are given as a function of t, i.e. $s(t)$ and $v(t)$. In most cases however, we will know the acceleration and then, the velocity and the position will have to be determined from the above expressions by integration.

Determining the velocity from the acceleration

From $a(t)$

If the acceleration is given as a function of t, $a(t)$, then the velocity can be determined by simple integration of equation (2),

$$v(t) = v_0 + \int_{t_0}^{t} a(t)\, dt \ . \tag{3}$$

Here, v_0 is the velocity at time t_0, which is determined by the initial conditions.

From $a(v)$

If the acceleration is given as a function of velocity $a(v)$, then, we can still use equation (2), but in this case we will solve for the time as a function of velocity,

$$t(v) = t_0 + \int_{v_0}^{v} \frac{dv}{a(v)} \ . \tag{4}$$

Once the relationship $t(v)$ has been obtained, we can, in principle, solve for the velocity to obtain $v(t)$. A typical example in which the acceleration is known as a function of velocity is when aerodynamic drag forces are present. Drag forces cause an acceleration which opposes the motion and is typically of the form $a(v) \propto v^2$ (the sign "$\propto$" means *proportional to*, that is, $a(v) = \kappa v^2$ for some κ, which is not a function of velocity).

From $a(s)$

When the acceleration is given as a function of s then, we need to use a combination of equations (1) and (2), to solve the problem. From $a\,dt = dv$ and $v\,dt = ds$ we eliminate dt and write

$$a\,ds = v\,dv \ . \tag{5}$$

This equation can now be used to determine v as a function of s,

$$v^2(s) = v_0^2 + 2\int_{s_0}^{s} a(s)\,ds \ . \tag{6}$$

where, v_0, is the velocity of the particle at point s_0. Here, we have used the fact that,

$$\int_{v_0}^{v} v\,dv = \int_{v_0}^{v} d(\frac{v^2}{2}) = \frac{v^{\,2}}{2} - \frac{v_0^{\,2}}{2} \ .$$

A classical example of an acceleration dependent on the spatial coordinate s, is that induced by a deformed linear spring. In this case, the acceleration is of the form $a(s) \propto s$.

Of course, when the acceleration is constant, any of the above expressions (3, 4, 6), can be employed. In this case we obtain,

$$v = v_0 + a(t - t_0), \quad \text{or} \quad v^2 = v_0^2 + 2a(s - s_0) \ .$$

Determining the position from the velocity

Once we know the velocity, the position can be found by integrating equation (1). Thus, when the velocity is known as a function of time we have,

$$s = s_0 + \int_{t_0}^{t} v(t)\,dt \ . \tag{7}$$

If the velocity is known as a function of position, then

$$t = t_0 + \int_{s_0}^{s} \frac{ds}{v(s)} \ . \tag{8}$$

Here, s_0 is the position at time t_0.

It is worth pointing out that equation (5), can also be used to derive an expression for $v(s)$, given $a(v)$,

$$s - s_0 = \int_{s_0}^{s} ds = \int_{v_0}^{v} \frac{v}{a(v)}\,dv \ . \tag{9}$$

This equation can be used whenever equation (4) is applicable and gives $v(s)$ instead of $t(v)$. For the case of constant acceleration, either of equations (7, 8), can be used to obtain,

$$s = s_0 + v_0(t - t_0) + \frac{1}{2}a(t - t_0)^2 \ .$$

In many practical situations, it may not be possible to carry out the above integrations analytically in which case, numerical integration is required. Usually, numerical integration will also be required when either the velocity or the acceleration depend on more than one variable, i.e. $v(s, t)$, or, $a(s, v)$.

Example ***Free falling body***

Consider an air-dropped payload starting from rest. The acceleration is a combination of gravity and air drag and has the form

$$a = g - 0.002725v^2 \, \mathrm{m/s} \ .$$

Here $g = 9.81\mathrm{m/s}^2$, is the acceleration due to gravity and v is the downward velocity. It is clear from this expression that initially the acceleration will be g. Therefore, the velocity will start to increase and keep on increasing until $a = 0$, at which point the velocity will stay constant. This velocity is called the *terminal velocity*, and in our case is given by,

$$0 = g - 0.002725\, v_f^2 \quad \text{or} \qquad v_f = 60\mathrm{m/s} \ .$$

The acceleration can be re-written introducing the terminal velocity as, $a = g(1 - (v/v_f)^2)$. In order to obtain an expression for the velocity as a function of time, we use expression (4), and write

$$t = \frac{1}{g}\int_0^v \frac{dv}{1 - (v/v_f)^2} = \frac{1}{g}\int_0^v \left(\frac{1/2}{1 + (v/v_f)} + \frac{1/2}{1 - (v/v_f)}\right) d\,v = \frac{v_f}{2g}\ln\left(\frac{v - v_f}{v + v_f}\right) \quad .$$

Solving for v we obtain,

$$v = v_f\,\frac{e^{2gt/v_f} - 1}{e^{2gt/v_f} + 1}\ \mathrm{m/s} \ .$$

We can easily verify that for large t, $v = v_f$. We can also find out how long does it take for the payload to reach, say, 95% of the terminal velocity,

$$t = \frac{2v_f}{g}\ln\frac{1.95}{0.05} = 11.21\mathrm{s} \ .$$

To obtain an expression for the velocity as a function of the travelled distance we can use expression (9) and write

$$s = \frac{1}{g}\int_0^v \frac{v\,dv}{1 - (v/v_f)^2} = -\frac{v_f^2}{2g}\ln(1 - (v/v_f)^2) \ .$$

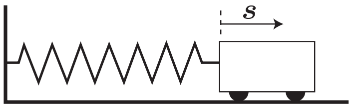
Solving for v we obtain

$$v = v_f\sqrt{1 - e^{-2gs/v_f^2}}\ \mathrm{m/s} \ .$$

We see that for, say, $v = 0.95v_f$, $s = 427.57$m. This is the distance travelled by the payload in 11.21s, which can be compared with the distance that would be travelled in the same time if we were to neglect air resistance, $s_{no\ drag} = gt^2/2 = 615.75$m.

Example **_Spring-mass system_**

Here, we consider a mass allowed to move without friction on a horizontal slider and subject to the force exerted by a linear spring. Initially the system is in equilibrium (no force on the spring) at $s = 0$. Suddenly, the mass is given a velocity v_0 and then the system is left free to oscillate. We know that the effect of the spring is to cause an acceleration to the body, opposing the motion, of the form $a = -\kappa s$, where $\kappa > 0$ is a constant.



Using equation (6), we have

$$v^2 = v_0^2 - \kappa s^2 \ .$$

The displacement can now be obtained using expression (8),

$$t = \int_0^s \frac{ds}{\sqrt{v_0^2 - \kappa s^2}} = \frac{1}{\sqrt{\kappa}} \arcsin \frac{\sqrt{\kappa} s}{v_0} \ ,$$

which gives,

$$s = \frac{v_0}{\sqrt{\kappa}} \sin \sqrt{\kappa} t \ .$$

Finally, the velocity as a function of time is simply, $v = v_0 \cos \sqrt{\kappa} t$. We recognize this motion as that of an undamped harmonic oscillator.

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

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