J. Peraire
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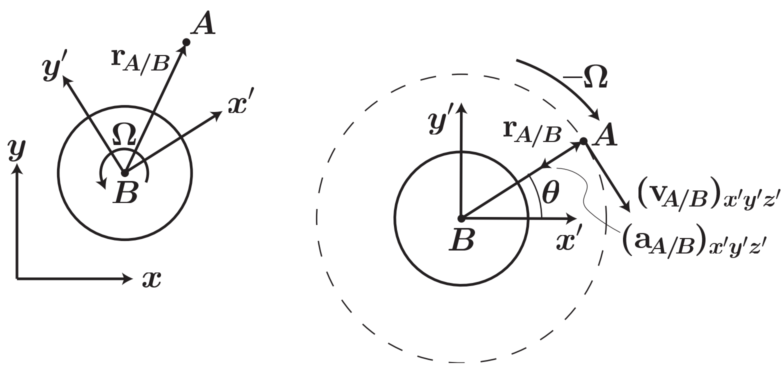
Lecture D12 - Relative Motion using Translating/Rotating Axes

In the previous lecture, we related the motion experienced by two observers in relative translational motion with respect to each other. In this lecture, we will extend this relation to our third type of observer. That is, observers who accelerate and rotate with respect to each other.

As a matter of illustration, let us consider a very simple situation, in which a particle at rest with respect to the fixed frame xy, i.e. $\boldsymbol{v}_A = \mathbf{0}$, $\boldsymbol{a}_A = \mathbf{0}$, is observed by an observer, B, who is standing at the center of a turn table. The table rotates with a constant angular velocity of Ω rad/s.



Assuming that the platform only rotates about its center and does not move translationally, the position of B will not change, and therefore $\boldsymbol{v}_B = \mathbf{0}$ and $\boldsymbol{a}_B = \mathbf{0}$. It is clear that relative to B, A will not be at rest. In fact, B will see A rotating about B with a constant angular velocity of $-\Omega$. Thus, we can easily calculate the motion of A as observed by B. Expressing the velocity and acceleration in local polar coordinates (rotating with $x'y'$), we will have

$$(\boldsymbol{v}_{A/B})_{x'y'} = (\dot{\boldsymbol{r}}_{A/B})_{x'y'} = -\Omega r_{A/B}\ \boldsymbol{e}_\theta\ ,$$

and

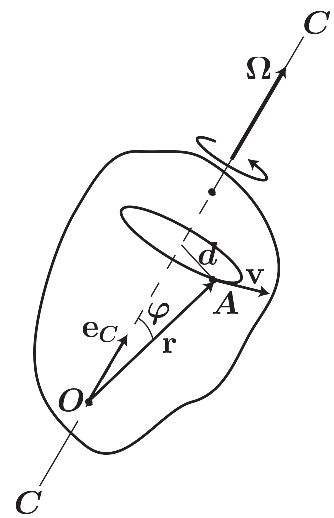
$$(\boldsymbol{a}_{A/B})_{x'y'} = (\dot{\boldsymbol{v}}_{A/B})_{x'y'} = -\frac{(v_{A/B})^2_{x'y'}}{r_{A/B}}\ \boldsymbol{e}_r = -\Omega^2 r_{A/B}\ \boldsymbol{e}_r\ .$$

The notation, $(\ \cdot\ )_{x'y'}$, is used to indicate the velocity, or acceleration, experienced by an observer that rotates with the axes $x'y'$. In other words, the time derivatives are taken assuming that the directions $x'y'$, as seen by a rotating observer, do not change. For illustration purposes, we note that $\boldsymbol{v}_{A/B} = \dot{\boldsymbol{r}}_{A/B} = 0$ and $\boldsymbol{a}_{A/B} = \dot{\boldsymbol{v}}_{A/B} = 0$. Here, $\boldsymbol{v}_{A/B}$ and $\boldsymbol{a}_{A/B}$ denote the relative velocity and acceleration of A with respect to B, experienced by a non-rotating observer (this was the situation considered in the last lecture). We could, in fact, write $(\boldsymbol{v}_{A/B})_{xy}$ and $(\boldsymbol{a}_{A/B})_{xy}$ instead of $\boldsymbol{v}_{A/B}$ and $\boldsymbol{a}_{A/B}$, but we will often omit the subscripts when the axes considered are fixed. Also, we note that $\boldsymbol{r}_{A/B} = (\boldsymbol{r}_{A/B})_{x'y'}$ since this is an instantaneous measurement and is not related to whether the axes rotate or not.

At this point, it should be clear that the expressions derived in the last lecture, which includes only translational relative motion, are not applicable when rotating observers are involved.

Note **_Angular velocity and angular acceleration vectors_**

In the two dimensional example presented above, we have dealt with the angular velocity as if it were a scalar quantity. Here, we will introduce the concept of angular velocity and acceleration vectors. We shall see that, especially in three dimensions, it makes a lot of sense to think of these quantities as vectors. Let us consider a rigid body, which is spinning about an axis $C - C$ with an angular velocity of Ω rad/s.



We consider a unit vector, $\boldsymbol{e}_C$, along the direction of the axis, and define the *angular velocity vector*, $\boldsymbol{\Omega}$, as a vector having magnitude Ω and direction $\boldsymbol{e}_C$. Thus,

$$\boldsymbol{\Omega} = \Omega \boldsymbol{e}_C \ .$$

Note that the convention between the direction of rotation and that of $\boldsymbol{\Omega}$ is determined by the right hand rule. If the body were to rotate in the direction opposite to that shown in the diagram, then we would simply have $\boldsymbol{\Omega} = -\Omega \boldsymbol{e}_C$.

The angular velocity vector is useful to express the velocity, due to rotation, of any point A in the rigid body. Let $\boldsymbol{r}$ be the position vector of A relative to an origin point, O, located on the axis of rotation. It turns out that the velocity of A, $\boldsymbol{v}$, can be simply expressed as

$$\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{r} \ . \tag{1}$$

First, we note that, when the body spins, A describes a circular trajectory around the axis of radius $d = r \sin\varphi$, where φ is the angle between $\boldsymbol{r}$ and the axis of rotation. Since the body is spinning at a rate of Ω rad/s, the magnitude of the velocity vector will be $v = \Omega d = \Omega r \sin\varphi$. The direction of the velocity vector

$\boldsymbol{v}$ will be tangent to the trajectory at A, which means that it will be perpendicular to $\boldsymbol{r}$ and $\boldsymbol{e}_C$. Thus, recalling the definition of the vector product, we can verify that the velocity given by (1) does indeed have the right direction, orientation and magnitude.

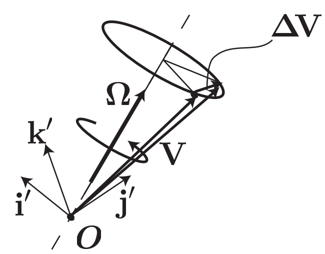
We also define the *angular acceleration* vector as,

$$\dot{\boldsymbol{\Omega}} = \frac{d\boldsymbol{\Omega}}{dt} \ .$$

Since the angular acceleration is the derivative of a vector, it will measure both the changes in magnitude and the changes in direction of the angular velocity vector.

Time derivative of a fixed vector in a rotating frame

We consider a reference frame $x'y'z'$ rotating with an angular velocity $\boldsymbol{\Omega}$ with respect to a fixed frame xyz. Let $\boldsymbol{V}$ be *any* vector, which is constant relative to the frame $x'y'z'$. That is, the vector components in the $x'y'z'$ frame do not change, and, as a consequence, $\boldsymbol{V}$ rotates as if it were rigidly attached to the frame.



In the absolute frame, the time derivative will be equal to

$$\frac{d\boldsymbol{V}}{dt} = \lim_{\Delta t \to 0} \frac{(\boldsymbol{V} + \Delta \boldsymbol{V}) - \boldsymbol{V}}{\Delta t} = \boldsymbol{\Omega} \times \boldsymbol{V} \ ,$$

which can be interpreted as the velocity of the tip of vector $\boldsymbol{V}$. The above expression applies to any vector which is rigidly attached to the frame $x'y'z'$. In particular, it applies to the unit vectors $\boldsymbol{i}'$, $\boldsymbol{j}'$ and $\boldsymbol{k}'$. Therefore, we have that

$$\begin{aligned} \frac{d\boldsymbol{i}'}{dt} &= \boldsymbol{\Omega} \times \boldsymbol{i}' \\ \frac{d\boldsymbol{j}'}{dt} &= \boldsymbol{\Omega} \times \boldsymbol{j}' \\ \frac{d\boldsymbol{k}'}{dt} &= \boldsymbol{\Omega} \times \boldsymbol{k}' \ . \end{aligned}$$

Time derivative of a vector in a rotating frame: Coriolis' theorem

Now let $\boldsymbol{V}$ be an arbitrary vector (e.g. velocity, magnetic field, force, etc.), which is allowed to change in both the fixed xyz frame and the rotating $x'y'z'$ frame. The vector $\boldsymbol{V}$ can be expressed, in the $x'y'z'$ frame, as,

$$\boldsymbol{V} = V_{x'}\boldsymbol{i}' + V_{y'}\boldsymbol{j}' + V_{z'}\boldsymbol{k}' .$$

If we now consider the time derivative of $\boldsymbol{V}$, as seen by the fixed frame, we have

$$\begin{aligned} \dot{\boldsymbol{V}} = \frac{d\boldsymbol{V}}{dt} &= \dot{V}_{x'}\boldsymbol{i}' + \dot{V}_{y'}\boldsymbol{j}' + \dot{V}_{z'}\boldsymbol{k}' + V_{x'}\dot{\boldsymbol{i}}' + V_{y'}\dot{\boldsymbol{j}}' + V_{z'}\dot{\boldsymbol{k}}' \\ &= (\dot{\boldsymbol{V}})_{x'y'z'} + \boldsymbol{\Omega} \times \boldsymbol{V} \ . \end{aligned} \tag{2}$$

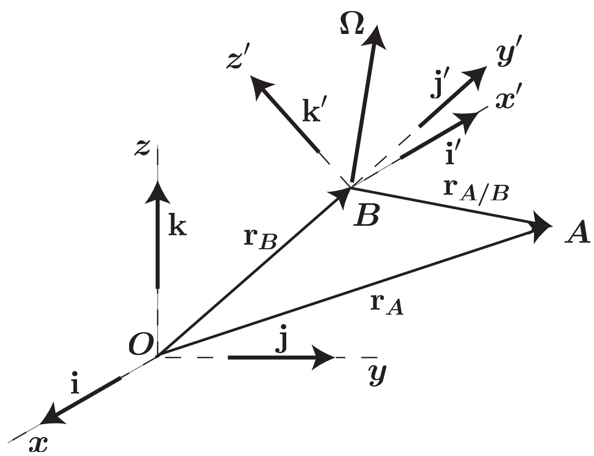
Here, $\dot{V}_{x'}\boldsymbol{i}' + \dot{V}_{y'}\boldsymbol{j}' + \dot{V}_{z'}\boldsymbol{k}' = (\dot{\boldsymbol{V}})_{x'y'z'}$ is the time derivative of the vector $\boldsymbol{V}$ *as seen by the rotating frame*. Hence, for this derivative, the vectors $\boldsymbol{i}'$, $\boldsymbol{j}'$ and $\boldsymbol{k}'$ remain unchanged. The term $V_{x'}\dot{\boldsymbol{i}}' + V_{y'}\dot{\boldsymbol{i}}' + V_{z'}\dot{\boldsymbol{i}}' = V_{x'}(\boldsymbol{\Omega} \times \boldsymbol{i}') + V_{y'}(\boldsymbol{\Omega} \times \boldsymbol{j}') + V_{z'}(\boldsymbol{\Omega} \times \boldsymbol{k}') = \boldsymbol{\Omega} \times \boldsymbol{V}$ is the change in $\boldsymbol{V}$ due to the rotation. The expression (2) above is known as *Coriolis' theorem*. Given an arbirary vector, it relates the derivative of that vector as seen by a fixed frame with the derivative of the same vector as seen by a rotating frame. Symbolically, we can write,

$$(\dot{\ \ })_{xyz} = (\dot{\ \ })_{x'y'z'} + \boldsymbol{\Omega} \times (\ \ ) \ .$$

We will often omit the notation $(\ \ )_{xyz}$ when the derivative is taken with respect to the fixed frame.

Relative Motion using translating/rotating axes

Here, we consider the relationship between the motion seen by an observer B that may be accelerating as well as rotating, and the motion seen by a a fixed observer O. Let $\boldsymbol{a}_B$ be the acceleration of B with respect to O, and let $\boldsymbol{\Omega}$ and $\dot{\boldsymbol{\Omega}}$ denote the angular velocity and angular acceleration, respectively, of the frame $x'y'z'$ rigidly attached to B. The vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the unit vectors corresponding to the fixed frame xyz, and $\boldsymbol{i}'$, $\boldsymbol{j}'$ and $\boldsymbol{k}'$ are the unit vectors corresponding to the rotating frame $x'y'z'$.



Position vector

The position vector of A with respect to the fixed frame can be written as,

$$\boldsymbol{r}_A = \boldsymbol{r}_B + \boldsymbol{r}_{A/B} \ . \tag{3}$$

Note that this is an instantaneous concept, and therefore it is immaterial whether the observer B is rotating or not. In other words, the above expression is valid at any given instant. Of course, if we choose to express $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ in xyz components, and $\boldsymbol{r}_{A/B}$ in $x'y'z'$ components, then we will have to make sure that the proper coordinate transformation is done before the components of the vectors are added. Also, since the orientation of $x'y'z'$ changes in time, this transformation will depend on the instant considered.

Velocity vector

Differentiating (3) with respect to time, we have,

$$\begin{aligned} \boldsymbol{v}_A = \dot{\boldsymbol{v}}_A &= \dot{\boldsymbol{r}}_B + (\dot{\boldsymbol{r}}_{A/B})_{x'y'z'} + \boldsymbol{\Omega} \times \boldsymbol{r}_{A/B} \\ &= \boldsymbol{v}_B + (\boldsymbol{v}_{A/B})_{x'y'z'} + \boldsymbol{\Omega} \times \boldsymbol{r}_{A/B} \ . \end{aligned} \tag{4}$$

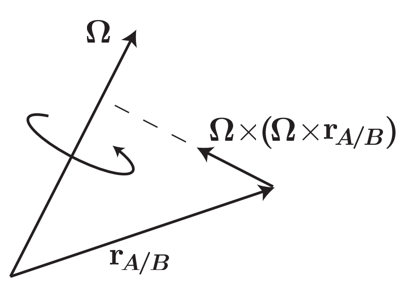
Here, we have used Coriolis' theorem (2) to calculate the derivative of $\boldsymbol{r}_{A/B}$, i.e. $\dot{\boldsymbol{r}}_{A/B} = (\dot{\boldsymbol{r}}_{A/B})_{x'y'z'} + \boldsymbol{\Omega} \times \boldsymbol{r}_{A/B}$. In the above expression, $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ are the velocities of A and B, respectively, relative to the fixed frame. The term $(\boldsymbol{v}_{A/B})_{x'y'z'}$ is the velocity of A measured by the rotating observer, B. Finally, $\boldsymbol{\Omega}$ is the angular velocity of the rotating frame, and $\boldsymbol{r}_{A/B}$ is the relative position vector of A with respect to B.

Acceleration vector

Differentiating (4) once again, and making use of Coriolis' theorem, e.g. $d(\boldsymbol{v}_{A/B})_{x'y'z'}/dt = (\dot{\boldsymbol{v}}_{A/B})_{x'y'z'} + \boldsymbol{\Omega} \times (\boldsymbol{v}_{A/B})_{x'y'z'}$, we obtain the following expression for the acceleration,

$$\begin{aligned} \boldsymbol{a}_A = \dot{\boldsymbol{v}}_A &= \dot{\boldsymbol{v}}_B + (\dot{\boldsymbol{v}}_{A/B})_{x'y'z'} + \boldsymbol{\Omega} \times (\boldsymbol{v}_{A/B})_{x'y'z'} \\ &+ \dot{\boldsymbol{\Omega}} \times \boldsymbol{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{v}_{A/B})_{x'y'z'} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_{A/B}) \\ &= \boldsymbol{a}_B + (\boldsymbol{a}_{A/B})_{x'y'z'} + 2\boldsymbol{\Omega} \times (\boldsymbol{v}_{A/B})_{x'y'z'} + \dot{\boldsymbol{\Omega}} \times \boldsymbol{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_{A/B}) \ . \end{aligned} \tag{5}$$

Here, $\boldsymbol{a}_A$ and $\boldsymbol{a}_B$ are the accelerations of A and B, respectively, observed by xyz. The term $(\boldsymbol{a}_{A/B})_{x'y'z'}$ is the acceleration of A measured by an observer B that rotates with the axes $x'y'z'$. The term $2\boldsymbol{\Omega} \times (\boldsymbol{v}_{A/B})_{x'y'z'}$ is called Coriolis' acceleration. The term $\dot{\boldsymbol{\Omega}} \times \boldsymbol{r}_{A/B}$ is due to the change in $\boldsymbol{\Omega}$, and, finally, the term $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_{A/B})$ is called centripetal acceleration. It can be easily seen that this acceleration always points towards the axis of rotation, and is orthogonal to $\boldsymbol{\Omega}$ (when trying to show this, you may find the vector identity, $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$, introduced in the Review Notes on Vectors, applied to $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_{A/B}) = (\boldsymbol{\Omega} \cdot \boldsymbol{r}_{A/B})\boldsymbol{\Omega} - (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega})\boldsymbol{r}_{A/B}$, useful).



Note — $\dot{\boldsymbol{\Omega}}$ **vs.** $(\dot{\boldsymbol{\Omega}})_{x'y'z'}$

When writing the angular acceleration vector, $\dot{\boldsymbol{\Omega}}$, we did not specify whether the derivative was taken with respect to the fixed observer, O, or with respect to the rotating observer, B. This may seem a little bit sloppy at first, but, in fact, it turns out that it does not really matter. The time derivatives of vectors which are parallel to $\boldsymbol{\Omega}$ are the same for both observers. This can be easily seen if we go back to Coriolis' theorem (2) and apply it to $\boldsymbol{\Omega}$. That is,

$$\dot{\boldsymbol{\Omega}} = (\dot{\boldsymbol{\Omega}})_{x'y'z'} + \boldsymbol{\Omega} \times \boldsymbol{\Omega} = (\dot{\boldsymbol{\Omega}})_{x'y'z'} \ .$$

Note — **Non-rotational observers**

We note that, for the case where $\boldsymbol{\Omega} = \dot{\boldsymbol{\Omega}} = \mathbf{0}$, the $x'y'z'$ axes only have translational motion relative to the fixed axes xyz, and therefore the equations (3), (4) and (5) reduce to those introduced in the previous lecture. Namely,

$$\begin{aligned} \boldsymbol{r}_A &= \boldsymbol{r}_B + \boldsymbol{r}_{A/B} \\ \boldsymbol{v}_A &= \boldsymbol{v}_B + \boldsymbol{v}_{A/B} \\ \boldsymbol{a}_A &= \boldsymbol{a}_B + \boldsymbol{a}_{A/B} \ . \end{aligned}$$

Note also that, since the axes $x'y'z'$ do not rotate, we do not need to use the notation $(\ \cdot\ )_{x'y'z'}$.

We can now go back to our introductory example and verify that expressions (4) and (5) are indeed consistent with the observations made earlier. We have $\boldsymbol{v}_A = \boldsymbol{v}_B = \mathbf{0}$, and $(\boldsymbol{v}_{A/B})_{x'y'z'} = -\Omega r_{A/B} \boldsymbol{e}_\theta$. Thus, equation (4) reduces to

$$\mathbf{0} \ = \ -\Omega r_{A/B} \boldsymbol{e}_\theta + \Omega \boldsymbol{k} \times r_{A/B} \boldsymbol{e}_r \ ,$$

which is clearly satisfied since $\boldsymbol{k} \times \boldsymbol{e}_r = \boldsymbol{e}_\theta$. For the acceleration equation (5), we have $\boldsymbol{a}_A = \boldsymbol{a}_B = \mathbf{0}$, $(\boldsymbol{v}_{A/B})_{x'y'z'} = -\Omega r_{A/B} \boldsymbol{e}_\theta$, $(\boldsymbol{a}_{A/B})_{x'y'z'} = -\Omega^2 r_{A/B} \boldsymbol{e}_r$, and $\dot{\boldsymbol{\Omega}} = \mathbf{0}$. Thus,

$$\mathbf{0} \ = \ -\Omega^2 r_{A/B} \boldsymbol{e}_r + 2\Omega \boldsymbol{k} \times (-\Omega r_{A/B} \boldsymbol{e}_\theta) + \Omega \boldsymbol{k} \times (\Omega \boldsymbol{k} \times r_{A/B} \boldsymbol{e}_r) \ ,$$

which is also satisfied since $\boldsymbol{k} \times \boldsymbol{e}_\theta = -\boldsymbol{e}_r$ and $\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{e}_r) = -\boldsymbol{e}_r$.

Note — **Velocity and acceleration of in a rotating rigid body**

We have seen (see earlier note in this lecture) that the velocity of a point A in a rigid body that is rotating with an angular velocity $\boldsymbol{\Omega}$ is simply

$$\boldsymbol{v} = \boldsymbol{v}_B + \boldsymbol{\Omega} \times \boldsymbol{r} \ ,$$

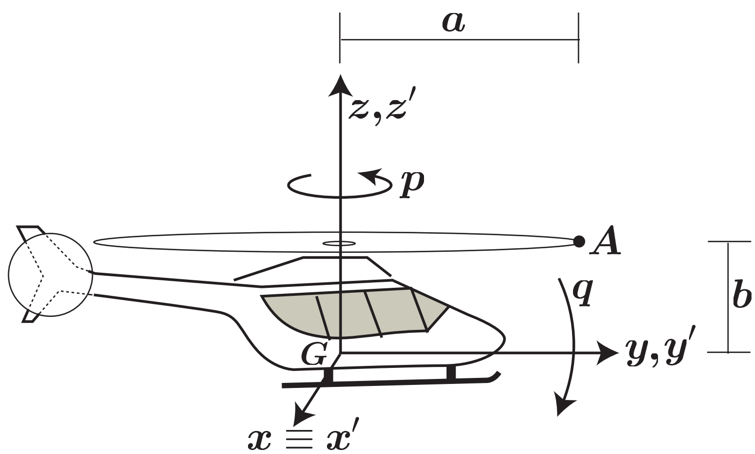
where $\boldsymbol{r}$ is the position vector of A with respect to point B, and $\boldsymbol{v}_B$ is the acceleration of point B. In order to determine the acceleration, we simply use expression (5). After setting all the appropriate to terms to zero, we get,

$$\boldsymbol{a} = \boldsymbol{a}_B + \dot{\boldsymbol{\Omega}} \times \boldsymbol{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) \ .$$

The second term is due to the change in angular velocity, whereas the third term is the centripetal acceleration due to the body's rotation.

Example **Helicopter Blades**

We want to determine the instantaneous velocity and acceleration of a point A, which is located at the tip of the blade of the helicopter. The blades are rotating with an angular velocity p, and, at the same time, the helicopter is pitching downwards at an angular velocity q. At the instant considered, the velocity and acceleration of the center of mass of the helicopter, G, are zero.



We will solve this problem using two approaches:

First approach :

In this approach, we consider a fixed set of axes, xyz, and a set of axes attached to the body of the helicopter $x'y'z'$. The origin of both axes is at the center of mass, G. Taking $B \equiv G$ in (4), we write

$$\boldsymbol{v}_A = \boldsymbol{v}_G + (\boldsymbol{v}_{A/G})_{x'y'z'} + \boldsymbol{\Omega} \times \boldsymbol{r}_{A/G}.$$

Here, $\boldsymbol{v}_G = \boldsymbol{0}$ since the center of mass is at rest. The velocity observed by the $x'y'z'$ frame is simply $(\boldsymbol{v}_{A/G})_{x'y'z'} = p\boldsymbol{k} \times \boldsymbol{r}_{A/G} = p\boldsymbol{k} \times (a\boldsymbol{j} + b\boldsymbol{k}) = -pa\boldsymbol{i}$. The angular velocity of the frame $x'y'z'$ is clearly the pitch angular velocity, thus $\boldsymbol{\Omega} = -q\boldsymbol{i}$. Finally, we have

$$\boldsymbol{v}_A = -pa\boldsymbol{i} + (-q\boldsymbol{i}) \times (a\boldsymbol{j} + b\boldsymbol{k}) = -pa\boldsymbol{i} + qb\boldsymbol{j} - qa\boldsymbol{k} \ .$$

For the acceleration, we use expression (5) and write

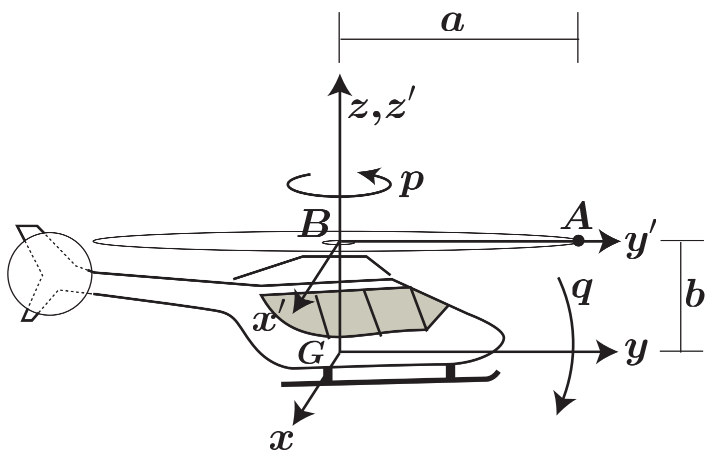
$$\boldsymbol{a}_A = \boldsymbol{a}_G + (\boldsymbol{a}_{A/G})_{x'y'z'} + 2\boldsymbol{\Omega} \times (\boldsymbol{v}_{A/G})_{x'y'z'} + \dot{\boldsymbol{\Omega}} \times \boldsymbol{r}_{A/G} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_{A/G}) \ .$$

Here, $\boldsymbol{a}_G = \boldsymbol{0}$ because the center of mass is not accelerating. Relative the the axes $x'y'z'$, point A describes a circular motion at constant speed. Therefore, the only observed acceleration will be the centripetal acceleration. The magnitude of the centripetal acceleration is simply the speed divided by the radius of curvature, and the direction is parallel to the negative y axis. Thus, $(\boldsymbol{a}_{A/G})_{x'y'z'} = -p^2 a\boldsymbol{j}$. The angular acceleration of $x'y'z'$ with respect to xyz is zero because the aircraft is pitching at a constant angular velocity, i.e., $\dot{\boldsymbol{\Omega}} = \boldsymbol{0}$. Finally, we have

$$\boldsymbol{a}_A = -p^2 a\boldsymbol{j} + 2(-q\boldsymbol{i}) \times (-pa\boldsymbol{i}) + (-q\boldsymbol{i}) \times (-q\boldsymbol{i} \times (a\boldsymbol{j} + b\boldsymbol{k})) = -(p^2 + q^2)a\boldsymbol{j} - q^2 b\boldsymbol{k} \ .$$

Second approach :

Here, we consider the same fixed set of axes xyz, but the moving axes $x'y'z'$ are now rigidly attached to the blades. Also, the origin of the moving axis is now taken at point B in the diagram.



The relative angular velocity of $x'y'z'$ with respect to xyz is now the sum of the pitch angular velocity $\boldsymbol{\Omega}_p = -q\boldsymbol{i}$, and the spin angular velocity $\boldsymbol{\Omega}_s = p\boldsymbol{k}$. Thus, we have

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_p + \boldsymbol{\Omega}_s = -q\boldsymbol{i} + p\boldsymbol{k} \ .$$

The expression for the angular acceleration follows from considering the derivative of the above expression. It is clear that the direction of $\boldsymbol{\Omega}_p$ is constant and its magnitude is also constant. Therefore, we have $\dot{\boldsymbol{\Omega}}_p = \boldsymbol{0}$. The situation with $\boldsymbol{\Omega}_s$ is different. The magnitude of $\boldsymbol{\Omega}_s$ is constant, but its direction changes. We can use Coriolis' theorem, (2), to compute its derivative. In order to do that, we consider a set of axes in which $\boldsymbol{\Omega}_s$ remains constant. For instance, we can use the axes $x'y'z'$ (the axes $x'y'z'$ of the first approach will also be a good choice). We have,

$$\dot{\boldsymbol{\Omega}}_s = (\dot{\boldsymbol{\Omega}}_s)_{x'y'z'} + \boldsymbol{\Omega} \times \boldsymbol{\Omega}_s \ .$$

Here, $(\dot{\boldsymbol{\Omega}}_s)_{x'y'z'} = \mathbf{0}$. Therefore, we have

$$\dot{\boldsymbol{\Omega}}_s = (\boldsymbol{\Omega}_p + \boldsymbol{\Omega}_s) \times \boldsymbol{\Omega}_s = \boldsymbol{\Omega}_p \times \boldsymbol{\Omega}_s = pq\boldsymbol{j} \ .$$

Now we can apply expressions (4) and (5) to calculate the velocity and acceleration. We note that $\boldsymbol{v}_B = qb\boldsymbol{j}$, and $\boldsymbol{a}_B = -q^2 b\boldsymbol{k}$. Also, $(\boldsymbol{v}_{A/B})_{x'y'z'} = \mathbf{0}$ and $(\boldsymbol{a}_{A/B})_{x'y'z'} = \mathbf{0}$ since, in the frame attached to the blades, point A is not moving. Setting $\boldsymbol{r}_{A/B} = a\boldsymbol{j}$, we have,

$$\boldsymbol{v}_A = qb\boldsymbol{j} + (-q\boldsymbol{i} + p\boldsymbol{k}) \times (a\boldsymbol{j}) = -pa\boldsymbol{i} - qb\boldsymbol{j} - qa\boldsymbol{k} \ ,$$

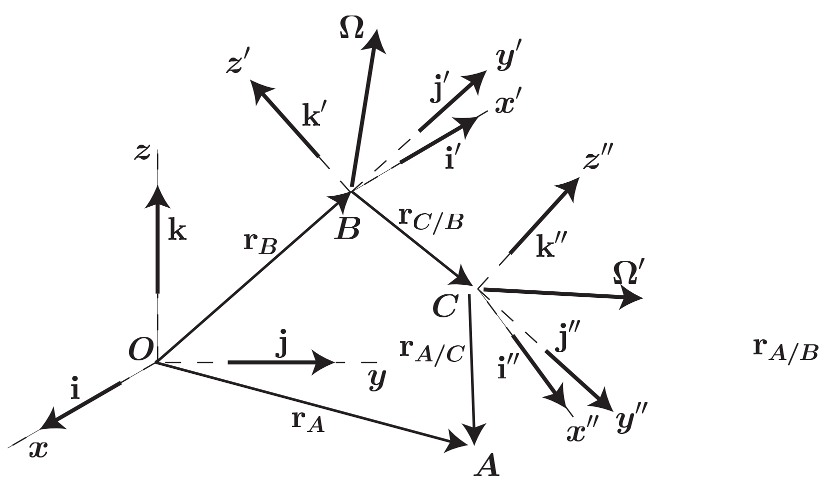
and

$$\boldsymbol{a}_A = (-q^2 b\boldsymbol{k}) + (pq\boldsymbol{j}) \times (a\boldsymbol{j}) + (-q\boldsymbol{i} + p\boldsymbol{k}) \times ((-q\boldsymbol{i} + p\boldsymbol{k}) \times (a\boldsymbol{j})) = -(p^2 + q^2)a\boldsymbol{j} - q^2 b\boldsymbol{k} \ .$$

Multiple observers

It should be noted that in all the above derivations we have assumed that O was fixed, and that B was translating and rotating relative to O. It turns out that the above derivations are still valid if O moves, possibly with acceleration and rotation, provided that $\boldsymbol{\Omega}$ is the difference between the angular velocities of B and O.

Realizing this allows us to extend the above expressions to the situation in which we have multiple observers.



In particular, if we consider the expression (4) for the velocity, the term $(\boldsymbol{v}_{A/B})_{x'y'z'}$ can be computed with the aid of a third observer C, using axes $x''y''z''$, which has a velocity $\boldsymbol{v}_{C/B}$ and angular velocity $\boldsymbol{\Omega}'$ with respect to $x'y'z'$. Thus, we can write

$$(\boldsymbol{v}_{A/B})_{x'y'z'} = (\boldsymbol{v}_{C/B})_{x'y'z'} + (\boldsymbol{v}_{A/C})_{x''y''z''} + (\boldsymbol{\Omega}')_{x'y'z'} \times \boldsymbol{r}_{A/C} \ ,$$

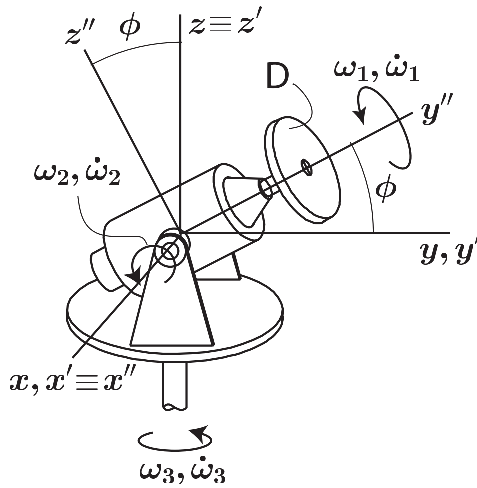
which, combined with (4), gives,

$$\begin{aligned} \boldsymbol{v}_A &= \boldsymbol{v}_B + (\boldsymbol{v}_{A/B})_{x'y'z'} + \boldsymbol{\Omega} \times \boldsymbol{r}_{A/B} \\ &= \boldsymbol{v}_B + (\boldsymbol{v}_{C/B})_{x'y'z'} + (\boldsymbol{v}_{A/C})_{x''y''z''} + (\boldsymbol{\Omega}')_{x'y'z'} \times \boldsymbol{r}_{A/C} + \boldsymbol{\Omega} \times \boldsymbol{r}_{A/B} \ . \end{aligned}$$

An analogous expression could be derived for the acceleration involving three observers in relative motion with respect to each other. You may want to try this as an exercise.

Example **Multiple Observers**

In this example we illustrate a more systematic procedure for calculating the angular velocities and accelerations when several reference frames are involved. We want to determine the angular acceleration of the disc D as a function of the angular velocities and accelerations given in the diagram. The angle of $\boldsymbol{\omega}_1$ with the horizontal is ϕ.



We consider three sets of axes. Axes xyz are fixed. Axes $x'y'z'$ rotate with angular velocity $\omega_3\boldsymbol{k}$ with respect to xyz. Axes $x''y''z''$ rotate with angular velocity $\omega_2\boldsymbol{i}'$ with respect to $x'y'z'$. Finally, the disc rotates with angular velocity $\omega_1\boldsymbol{j}''$ with respect to the axes $x''y''z''$.

The angular velocity of the disc with respect to the fixed axes will be simply

$$\boldsymbol{\Omega} = \omega_2\boldsymbol{i}' + \omega_1\boldsymbol{j}'' + +\omega_3\boldsymbol{k} \ . \tag{6}$$

At the instant considered, $\boldsymbol{i}' = \boldsymbol{i}$ and $\boldsymbol{j}'' = \cos\phi\boldsymbol{j} + \sin\phi\boldsymbol{k}$. Therefore, we can also write,

$$\boldsymbol{\Omega} = \omega_2\boldsymbol{i} + \omega_1\cos\phi\boldsymbol{j} + (\omega_1\sin\phi + \omega_3)\boldsymbol{k} \ .$$

In order to calculate the angular acceleration of the disc with respect to the fixed axes xyz, we start from (6) and write,

$$\left(\frac{d\boldsymbol{\Omega}}{dt}\right)_{xyz} = \left(\frac{d}{dt}(\omega_2\boldsymbol{i}' + \omega_1\boldsymbol{j}'' + \omega_3\boldsymbol{k})\right)_{xyz} = \left(\frac{d}{dt}(\omega_2\boldsymbol{i}' + \omega_1\boldsymbol{j}'')\right)_{xyz} + \dot{\omega}_3\boldsymbol{k} \ .$$

Here, we have used the fact that $\boldsymbol{k}$ does not change with respect to the xyz axes and therefore only the magnitude of ω_3 changes. In order to calculate the time derivative of $\omega_2\boldsymbol{i}' + \omega_1\boldsymbol{j}''$ with respect to the inertial reference frame, we apply Coriolis' theorem (2). Since $x'y'z'$ rotates with angular velocity $\omega_3\boldsymbol{k}$ with respect to xyz, we write,

$$\left(\frac{d}{dt}(\omega_2\boldsymbol{i}' + \omega_1\boldsymbol{j}'')\right)_{xyz} = \left(\frac{d}{dt}(\omega_2\boldsymbol{i}' + \omega_1\boldsymbol{j}'')\right)_{x'y'z'} + \omega_3\boldsymbol{k} \times (\omega_2\boldsymbol{i}' + \omega_1\boldsymbol{j}'') \ .$$

In the $x'y'z'$ frame, $\boldsymbol{i}'$ does not change direction. Therefore, we can write

$$\left(\frac{d}{dt}(\omega_2\boldsymbol{i}' + \omega_1\boldsymbol{j}'')\right)_{xyz} = \dot{\omega}_2\boldsymbol{i}' + \left(\frac{d}{dt}(\omega_1\boldsymbol{j}'')\right)_{x'y'z'} + \omega_3\boldsymbol{k} \times (\omega_2\boldsymbol{i}' + \omega_1\boldsymbol{j}'') \ .$$

In order to evaluate the derivative of $\omega_1\boldsymbol{j}''$ with respect to the $x'y'z'$ frame we make use again of Coriolis' theorem (2), and write

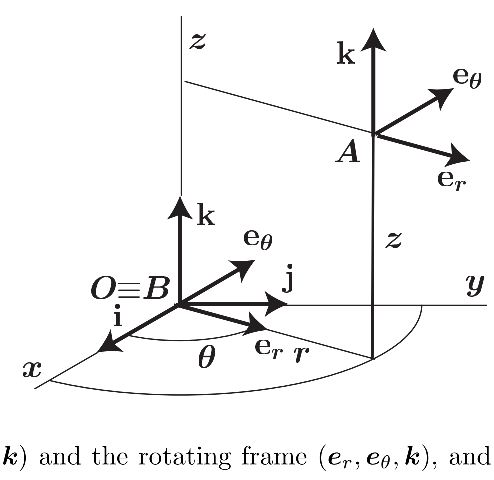
$$\left(\frac{d}{dt}(\omega_1\boldsymbol{j}'')\right)_{x'y'z'} = \left(\frac{d}{dt}(\omega_1\boldsymbol{j}'')\right)_{x''y''z''} + \omega_2\boldsymbol{i}' \times \omega_1\boldsymbol{j}'' = \dot{\omega}_1\boldsymbol{j}'' + \omega_2\boldsymbol{i}' \times \omega_1\boldsymbol{j}'' \ .$$

Here, we have used the fact that $x''y''z''$ rotates with angular velocity $\omega_2\boldsymbol{i}'$ with respect to $x'y'z'$, and the derivative of $\boldsymbol{j}''$ in the $x''y''z''$ reference frame is zero. Putting it all together, we have,

$$\begin{aligned}
\left(\frac{d\boldsymbol{\Omega}}{dt}\right)_{xyz} &= \dot{\omega}_2\boldsymbol{i}' + \dot{\omega}_1\boldsymbol{j}'' + \omega_2\boldsymbol{i}' \times \omega_1\boldsymbol{j}'' + \omega_3\boldsymbol{k} \times (\omega_2\boldsymbol{i}' + \omega_1\boldsymbol{j}'') + \dot{\omega}_3\boldsymbol{k} \\
&= \dot{\omega}_2\boldsymbol{i}' + \dot{\omega}_1\boldsymbol{j}'' + \omega_1\omega_2\boldsymbol{k}'' + \omega_2\omega_3\boldsymbol{j} - \omega_1\omega_3\cos\phi\ \boldsymbol{i} + \dot{\omega}_3\boldsymbol{k} \\
&= (\dot{\omega}_2 - \omega_1\omega_3\cos\phi)\ \boldsymbol{i} + (\dot{\omega}_1\cos\phi + \omega_2\omega_3 - \omega_1\omega_2\sin\phi)\ \boldsymbol{j} + (\dot{\omega}_3 + \dot{\omega}_1\sin\phi + \omega_1\omega_2\cos\phi)\ \boldsymbol{k}.
\end{aligned}$$

Example — ***Cylindrical coordinates***

Cylindrical (and spherical) coordinate systems can be regarded as particular cases of the expressions (3), (4) and (5).



Consider the fixed frame ($\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$) and the rotating frame ($\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{k}$), and consider two observers O and B whose positions coincide, but who refer to the fixed and the rotating frame, respectively. Thus, we have that

$\boldsymbol{r}_B = \boldsymbol{v}_B = \boldsymbol{a}_B = \boldsymbol{0}$. The angular velocity and acceleration of the rotating frame relative to the fixed frame are $\boldsymbol{\Omega} = \dot{\theta}\boldsymbol{k}$ and $\dot{\boldsymbol{\Omega}} = \ddot{\theta}\boldsymbol{k}$, respectively. Therefore, expressions (3), (4) and (5) become,

$$
\begin{aligned}
\boldsymbol{r}_A &= \boldsymbol{r}_{A/B} = r\boldsymbol{e}_r + z\boldsymbol{k} \\
\boldsymbol{v}_A &= (\boldsymbol{v}_{A/B})_{r\theta z} + \Omega \times \boldsymbol{r}_{A/B} = \dot{r}\boldsymbol{e}_r + \dot{z}\boldsymbol{k} + \dot{\theta}\boldsymbol{k} \times (r\boldsymbol{e}_r + z\boldsymbol{k}) = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + \dot{z}\boldsymbol{k} \\
\boldsymbol{a}_A &= (\boldsymbol{a}_{A/B})_{r\theta z} + 2\boldsymbol{\Omega} \times (\boldsymbol{v}_{A/B})_{r\theta z} + \dot{\boldsymbol{\Omega}} \times \boldsymbol{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_{A/B}) \\
&= (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta + \ddot{z}\boldsymbol{k} \ ,
\end{aligned}
$$

which agree with the previously derived expressions for position, velocity and acceleration in cylindrical coordinates. Here, $(\boldsymbol{a}_{A/B})_{r\theta k} = \ddot{r}\boldsymbol{e}_r + \ddot{z}\boldsymbol{k}$, $2\boldsymbol{\Omega} \times (\boldsymbol{v}_{A/B})_{r\theta k} = 2\dot{r}\dot{\theta}\boldsymbol{e}_\theta$, $\dot{\boldsymbol{\Omega}} \times \boldsymbol{r}_{A/B} = r\ddot{\theta}\boldsymbol{e}_\theta$, and $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_{A/B}) = -r\dot{\theta}^2\boldsymbol{e}_r$.

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

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