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Lecture D7 - Work and Energy

So far we have used Newton's second law $\boldsymbol{F} = m\boldsymbol{a}$ to establish the instantaneous relation between the sum of the forces acting on a particle and the acceleration of that particle. Once the acceleration is known, the velocity (or position) is obtained by integrating the expression of the acceleration (or velocity).
There are two situations in which the cumulative effects of unbalanced forces acting on a particle are of interest to us. These involve:

a) forces acting along the trajectory. In this case, integration of the forces with respect to the *displacement* leads to the principle of *work* and *energy*.

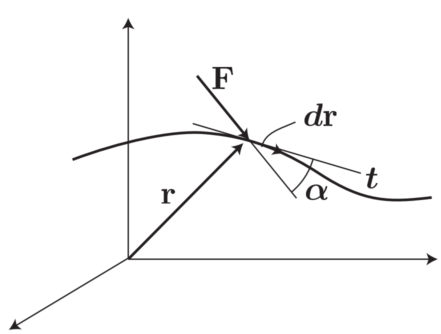
b) forces acting over a time interval. In this case, integration of the forces with respect to the *time* leads to the principle of *impulse* and *momentum*.

It turns out that in many situations, these integrations can be carried beforehand to produce equations that relate the velocities velocities at the initial and final integration points. In this way, the velocity can be obtained directly, thus making it unnecessary to solve for the acceleration. We shall see that these integrated forms of the equations of motion are very useful in the practical solution of dynamics problems.
In this lecture, we will concentrate on situation a), and consider the space integrated form of Newton's second law. We will defer the discussion of b), time integrated equations, to lecture D9.

Mechanical Work

Consider a force $\boldsymbol{F}$ acting on a particle that moves along a path. Let $\boldsymbol{r}$ be the position of the particle measured relative to the origin O. The work done by the force $\boldsymbol{F}$ when the particle moves an infinitesimal amount $d\boldsymbol{r}$ is defined as

$$dW = \boldsymbol{F} \cdot d\boldsymbol{r} \ . \qquad (1)$$



That is, the work done by the force $\boldsymbol{F}$, over an infinitesimal displacement $d\boldsymbol{r}$, is the *scalar* product of $\boldsymbol{F}$ and $d\boldsymbol{r}$. It follows that the work is a *scalar* quantity. Using the definition of the scalar product, we have that $dW = \boldsymbol{F} \cdot d\boldsymbol{r} = F ds \cos\alpha$, where ds is the modulus of $d\boldsymbol{r}$, and α is the angle between $\boldsymbol{F}$ and $d\boldsymbol{r}$. Since $d\boldsymbol{r}$ is parallel to the tangent vector to the path, $\boldsymbol{e}_t$, (i.e. $d\boldsymbol{r} = ds\, \boldsymbol{e}_t$), we have that $\boldsymbol{F} \cdot \boldsymbol{e}_t = F_t$. Thus,

$$dW = F_t\, ds \ , \tag{2}$$

which implies that only the tangential component of the force "does" work.

During a finite increment in which the particle moves from position $\boldsymbol{r}_1$ to position $\boldsymbol{r}_2$, the total work done by $\boldsymbol{F}$ is

$$W_{12} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{s_1}^{s_2} F_t\, ds \ . \tag{3}$$

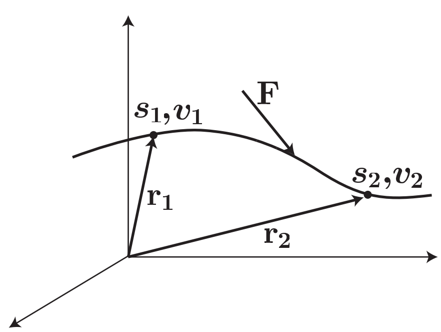
Here, s_1 and s_2 are the path coordinates corresponding to $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

Note ***Units of Work***

In the international system, SI, the unit of work is the *Joule* (J). We have that 1 J = 1 N · m. In the English system the unit of work is the ft-lb. We note that the units of work and moment are the same. It is customary to use ft-lb for work and lb-ft for moments to avoid confusion.

Principle of Work and Energy

We now a consider a particle moving along its path from point $\boldsymbol{r}_1$ to point $\boldsymbol{r}_2$. The path coordinates at points 1 and 2 are s_1 and s_2, and the corresponding velocity magnitudes v_1 and v_2.



If we start from (3) and use Newton's second law ($\boldsymbol{F} = m\boldsymbol{a}$) to express $F_t = ma_t$, we have

$$W_{12} = \int_{s_1}^{s_2} F_t\, ds = \int_{s_1}^{s_2} ma_t\, ds = \int_{v_1}^{v_2} mv\, dv = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \ . \tag{4}$$

Here, we have used the relationship $a_t\, ds = v\, dv$, which can be easily derived from $a_t = \dot{v}$ and $v = \dot{s}$ (see lecture D4).

Defining the *kinetic energy*[1], as

$$T = \frac{1}{2}mv^2 \ ,$$

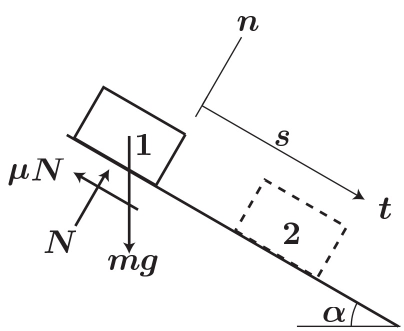
we have that,

$$W_{12} = T_2 - T_1 \qquad \text{or} \qquad T_1 + W_{12} = T_2 \ . \tag{5}$$

The above relationship is known as the *principle of work and energy*, and states that the mechanical work done on a particle is equal to the change in the kinetic energy of the particle.

Example — ***Block sliding down an incline***

A block is released from rest at the top of a ramp. The coefficient of kinetic friction between the surface of the ramp and the block is μ. We want to determine the velocity of the block as a function of the distance traveled on the ramp, s.



The forces on the block are: the weight, mg, the normal force, N, and, the friction force, μN. We have that $F_n = ma_n$ and $F_t = ma_t$. Since $F_n = N - mg\cos\alpha$ and $a_n = 0$, we have $N = mg\cos\alpha$. Thus, $F_t = mg\sin\alpha - \mu N = mg\sin\alpha - \mu mg\cos\alpha$, which is constant. If we apply the principle of work and energy between the position (1), when the block is at rest at the top of the ramp, and the position (2), when the block has travelled a distance s, we have $T_1 = 0$, $T_2 = (mv^2)/2$, and the work done by F_t is simply $W_{12} = F_t s$. Thus,

$$T_1 + W_{12} = T_2 \ , \qquad \text{or,} \qquad mg(\sin\alpha - \mu\cos\alpha) = \frac{1}{2}mv^2 \ .$$

From which we obtain, for the velocity,

$$v = \sqrt{2g(\sin\alpha - \mu\cos\alpha)s} \ .$$

We make two observations: first, the normal force, N, does no work since it is, at all times, perpendicular to the path, and second, we have obtained the velocity of the block directly without having to carry out any integrations. Note that an alternative, longer approach would have been to directly use $\boldsymbol{F} = m\boldsymbol{a}$, and integrate the corresponding expression for the acceleration.

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[1] The use of T to denote the kinetic energy, instead of K, is customary in dynamics textbooks

Note **Alternative expressions for dW**

We have seen in expression (2) that a convenient set of coordinates to express dW are the tangential- normal-binormal coordinates. Alternative expressions can be derived for other coordinate systems. For instance, we can express $dW = \boldsymbol{F} \cdot d\boldsymbol{r}$ in:

cartesian coordinates,

$$dW = F_x\, dx + F_y\, dy + F_z\, dz \ ,$$

cylindrical (polar) coordinates,

$$dW = F_r\, dr + F_\theta\, r d\theta + F_z\, dz \ ,$$

or spherical coordinates,

$$dW = F_r\, dr + F_\theta\, r \cos\phi\, d\theta + F_\phi\, r d\phi \ .$$

As an illustration, let's calculate the work done by a constant force, such as that due to gravity. The force on a particle of mass m is given by $\boldsymbol{F} = -mg\boldsymbol{k}$. When the particle moves from position $\boldsymbol{r}_1 = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$ to position $\boldsymbol{r}_2 = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}$, work is done, and the work may be written as

$$W_{12} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{z_1}^{z_2} -mg\, dz = -mg(z_2 - z_1) \ .$$

Power

In many situations it is useful to consider the rate at which a device can deliver work. The work per unit time is called the *power*, P. Thus,

$$P = \frac{dW}{dt} = \boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{dt} = \boldsymbol{F} \cdot \boldsymbol{v} \ .$$

The unit of power in the SI system is the *Watt* (W). We have that 1 W = 1 J /s. In the English system the unit of power is the ft-lb/s. A common unit of power is also the *horse power* (hp), which is equivalent to 550 ft-lb/s, or 746 W.

Note **Efficiency**

The ratio of the power delivered *out of* a system, P_{out}, to the power delivered *in to* the system, P_{in}, is called the *efficiency*, e, of the system.

$$e = \frac{P_{out}}{P_{in}} \ .$$

This definition assumes that the energy into and out of the system flows continuously and is not retained within the system. The efficiency of any real machine is always less than unity since there is always some mechanical energy dissipated as heat due to friction forces.

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

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