

## Lecture D13 - Newton's Second Law for Non-Inertial Observers. Inertial Forces

### Inertial reference frames

In the previous lecture, we derived an expression that related the accelerations observed using two reference frames, A and B, which are in relative motion with respect to each other.

$$\mathbf{a}_A = \mathbf{a}_B + (\mathbf{a}_{A/B})_{x'y'z'} + 2\boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'} + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) . \quad (1)$$

Here,  $\mathbf{a}_A$  is the acceleration of particle A observed by one observer, and  $(\mathbf{a}_{A/B})_{x'y'z'}$  is the acceleration of *the same particle* observed by the other (moving) observer. It is clear that the acceleration of particle A will be different for each observer, unless all the other terms in the above expression are zero. This means that if one of the observers is inertial, the other observer will be inertial if and only if  $\dot{\boldsymbol{\Omega}} = \mathbf{0}$ ,  $\boldsymbol{\Omega} = \mathbf{0}$  and  $\mathbf{a}_B = \mathbf{0}$ . Thus, we conclude that,

- inertial frames can not rotate with respect to each other, i.e.,  $\boldsymbol{\Omega} = \mathbf{0}$  and  $\dot{\boldsymbol{\Omega}} = \mathbf{0}$ , and,
- inertial frames can not be accelerating with respect to each other, i.e.  $\mathbf{a}_B = \mathbf{0}$ .

Thus, inertial frames can only be at most in constant relative velocity with respect to each other. In practical terms, the closest that we are able to get to an inertial frame is one which is in constant relative velocity with respect to the most distant stars.

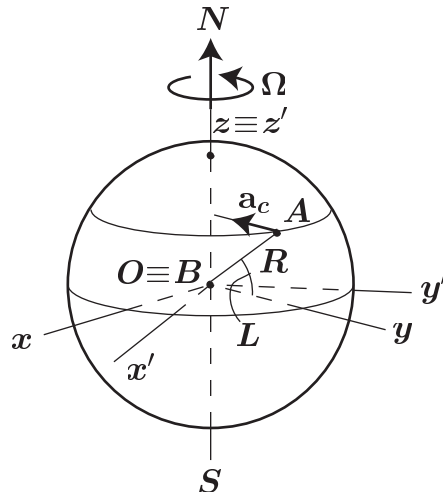
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#### Note

#### *The earth as an inertial reference frame*

Given that the earth is rotating about itself and at the same time is rotating about the sun, it is clear that the earth can not be an inertial reference frame. However, we shall see that, for many applications, the error made in assuming that the earth is an inertial reference frame is small.

We can easily estimate the effect of earth's rotation. Consider for instance two reference frames  $xyz$  and  $x'y'z'$ . The first frame is fixed and the second frame rotates with the earth.



In this case,  $\dot{\boldsymbol{\Omega}} = \mathbf{0}$ , since the earth rotates with a constant angular velocity, and  $\mathbf{a}_B = \mathbf{0}$ .

The centripetal acceleration,  $\mathbf{a}_c = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B})$ , will depend on the point considered and is directed towards the axis of rotation. The modulus is given by  $a_c = R\Omega^2 \cos L$ . For the earth,  $\Omega \approx 7.3 \times 10^{-5}$  rad/s,  $R = 20 \times 10^6$  ft, and, if we consider, for instance, a point located at a latitude of  $L = 40^\circ$ , then,

$$a_c \approx 0.08 \text{ft/s}^2 .$$

The Coriolis acceleration,  $\mathbf{a}_{cor} = 2\boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'}$ , depends on the velocity of  $A$  relative to the rotating earth and is zero if the point is not moving relative to the earth. On the other hand, for an aircraft flying in the east–west direction, at a speed of 250 m/s ( $\approx 718$  ft/s),  $\mathbf{a}_{cor}$  would be in the radial direction at  $A$  (local vertical) and pointing away from the center of the earth (upwards). The magnitude will be

$$a_{cor} \approx 0.10 \text{ft/s}^2 .$$

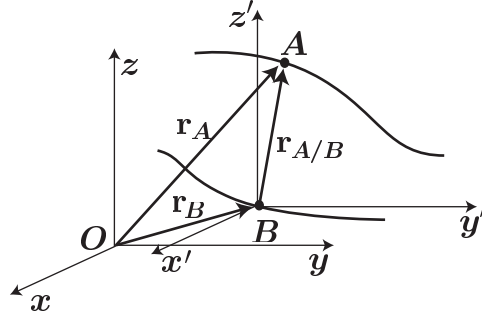
We see that these values, although not negligible in many situations, are still small when compared with the acceleration due to gravity of  $g = 32.2$  ft/s<sup>2</sup>.

## Inertial forces

We know that Newton's second law is only applicable when the motion is referred to inertial observers. This means that, before Newton's second law can be applied, the acceleration measured by a non-inertial observer needs to be transformed to an inertial acceleration. This transformation is achieved using the relative motion expressions derived in the previous lecture. An alternative approach is to extend Newton's second law to general non-inertial observers. This will lead to the concept of inertial forces. We shall see that, in many practical situations, it is convenient to work directly in the accelerated reference frame.

## Translating observers

We consider an inertial observer  $O$  and, to start, a translating (possibly accelerating) observer  $B$ .



Recall that the expression relating the accelerations of a particle  $A$  as observed by  $O$ ,  $\mathbf{a}_A$ , and as observed by  $B$ ,  $\mathbf{a}_{A/B}$ , is

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B} . \quad (2)$$

Since  $O$  is inertial, he or she will be able to verify Newton's second law. Thus, if  $m$  is the mass of particle  $A$ , we will have,

$$\mathbf{F} = m \mathbf{a}_A = m (\mathbf{a}_B + \mathbf{a}_{A/B}) , \quad (3)$$

where  $\mathbf{F}$  is the vector sum of all the forces acting on  $A$ . The above equation can be re-written as

$$\mathbf{F} - m \mathbf{a}_B = m \mathbf{a}_{A/B} , \quad (4)$$

and, defining the *inertial force*,  $\mathbf{F}_{inertial}$ , as

$$\mathbf{F}_{inertial} = -m \mathbf{a}_B , \quad (5)$$

we can combine the external and inertial forces into the apparent force,  $\mathbf{F}_{apparent}$ , to obtain

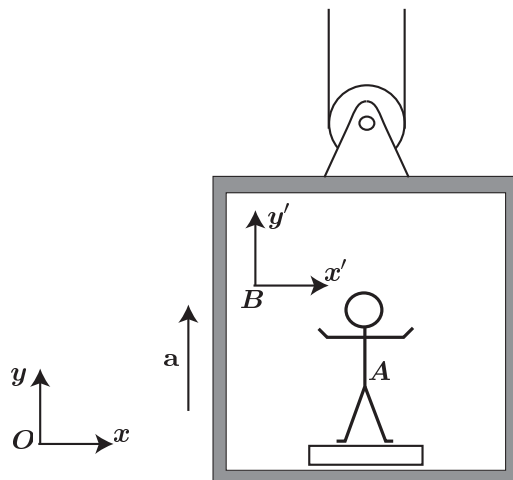
$$\mathbf{F}_{apparent} \equiv \mathbf{F} + \mathbf{F}_{inertial} = m \mathbf{a}_{A/B} .$$

Thus, we see that, if in addition to the external force, we include the inertial force, then observer  $B$ , will be able to verify a “modified” Newton's second law. That is, for observer  $B$ , the apparent force is equal to the mass times the observed acceleration. This concept is sometimes called D’Alambert’s principle, and the inertial force is sometimes referred to as the D’Alambert force.

Although inertial forces manifest themselves as real forces to non-inertial observers, they are sometimes called fictitious. They are called fictitious forces because they do not result from interactions with other bodies, like other forces (e.g. gravity, electromagnetic, contact, etc.). Fictitious forces “exist” because of the acceleration of the observer.

Note that when the acceleration of  $B$  with respect to  $O$ ,  $\mathbf{a}_B$ , is zero (i.e.  $B$  moves, at most, with a constant velocity with respect to  $O$ ), then  $\mathbf{a}_A = \mathbf{a}_{A/B}$ , which means that  $B$  is also inertial, and, hence, Newton's law is verified with respect to  $B$ .

A boy of weight  $W = mg$  is standing on a scale inside an elevator which is moving with an acceleration  $\mathbf{a}$ . We want to determine the reading on the scale.



This problem can be approached in two ways:

#### Inertial observer

Here, we look at the problem from the perspective of an inertial observer who is outside the elevator and sees the elevator passing by with an acceleration  $\mathbf{a}$ . For this observer, the sum of the forces on the boy will have to equal the mass of the boy times its inertial acceleration  $\mathbf{a}_A = \mathbf{a}$ . Thus, if  $\mathbf{R}$  is the reaction force that the scale exerts on the boy, we will have for the vertical direction,

$$-mg + R = ma ,$$

or,  $R = m(g + a)$ .

#### Accelerated observer

Here, we look at the problem from the perspective of an accelerated observer,  $B$ , who is inside the elevator. The acceleration of this observer with respect to the inertial observer  $O$ , is  $\mathbf{a}_B = \mathbf{a}$ . On the other hand, relative to  $B$ , the boy will not move. Thus, observer  $B$  will observe that  $\mathbf{a}_{A/B} = \mathbf{0}$ . We have that  $\mathbf{F} = (-mg + R)\mathbf{j}$  and  $\mathbf{F}_{inertial} = -m\mathbf{a}_B$ . Thus, using expression (4) and solving for the vertical direction,

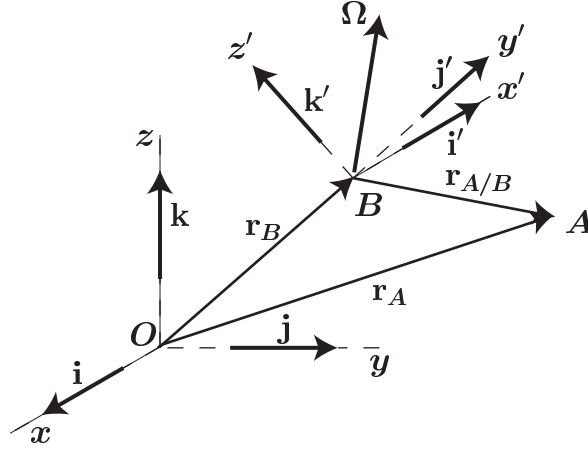
$$-mg + R - ma = 0,$$

which obviously gives  $R = m(g + a)$ .

We also see that, if the acceleration is downwards and equal to the gravity acceleration,  $\mathbf{a} = -g\mathbf{j}$  (i.e. free fall), then the reaction force,  $\mathbf{R}$ , will be zero, as expected.

## Translating/Rotating observers

In the more general situation in which the accelerated observer can also rotate, we can also define additional inertial forces so that Newton's second law is satisfied for the accelerating observer, but in this case, the inertial forces must account for the rotational effects.



Recall that the expression relating the acceleration of a particle  $A$ , observed by  $O$ ,  $\mathbf{a}_A$ , and observed by the translating/rotating observer,  $B$ ,  $(\mathbf{a}_{A/B})_{x'y'z'}$ , is

$$\mathbf{a}_A = \mathbf{a}_B + (\mathbf{a}_{A/B})_{x'y'z'} + 2\boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'} + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) . \quad (6)$$

Since  $O$  is inertial, he or she will be able to verify Newton's second law. Thus, if  $m$  is the mass of particle  $A$ , we will have,

$$\mathbf{F} = m\mathbf{a}_A , \quad (7)$$

where  $\mathbf{F}$  is the vector sum of all the forces acting on  $A$ . Inserting (6) into the above equation and rearranging the terms, we have

$$\mathbf{F} - m\mathbf{a}_B - 2m\boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'} - m\dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) = m(\mathbf{a}_{A/B})_{x'y'z'} . \quad (8)$$

If we define the inertial force,  $\mathbf{F}_{inertial}$ , as

$$\mathbf{F}_{inertial} = -m\mathbf{a}_B - 2m\boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'} - m\dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}), \quad (9)$$

then, we can write

$$\mathbf{F}_{apparent} \equiv \mathbf{F} + \mathbf{F}_{inertial} = m(\mathbf{a}_{A/B})_{x'y'z'} .$$

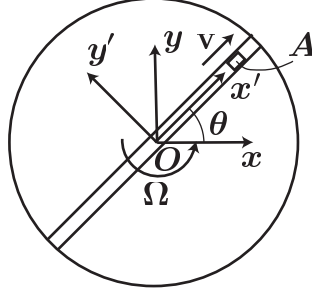
We know that the centripetal acceleration,  $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B})$ , points towards the axis of rotation. The inertial force associated with it,  $\mathbf{F}_c = -m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B})$ , is a force that points away from the axis of rotation, and, hence, it is sometimes called the centrifugal force. The inertial force for the general translating/rotating observer is comprised of various other terms which will be illustrated in the following lectures.

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**Example****Mass moving radially on a rotating table**

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We first consider a table rotating at uniform speed  $\Omega$  about its center  $O$ , and a point mass  $m$  at  $A$ , moving at constant speed  $v$  along the radial track attached to the table. For an inertial observer, the path of the point in space is a spiral curve.



To determine the force,  $\mathbf{R}$ , that the track exerts on  $m$ , we consider an inertial non-rotating observer at  $O$ , and an observer,  $B \equiv O$ , rotating with the table. From (6), noting that  $\dot{\mathbf{\Omega}} = \mathbf{0}$ , and  $(\mathbf{a}_{A/B})_{x'y'z'} = \mathbf{0}$ , we have,

$$\mathbf{R} - 2m\mathbf{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'} - m\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{A/B}) = \mathbf{0} ,$$

or,

$$\mathbf{R} = 2m\Omega v \mathbf{e}_\theta - m\Omega^2 r \mathbf{e}_r .$$

The circumferential force  $2m\Omega v$  is due to Coriolis' acceleration, and, in this case, it has a simple interpretation. When  $r$  increases, the circumferential velocity of  $m$ ,  $\Omega r$ , also increases, and the corresponding acceleration is provided by the circumferential Coriolis force.

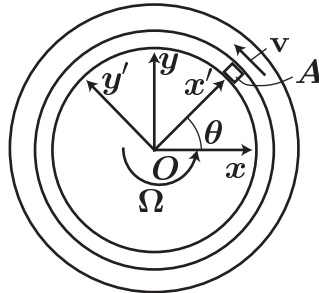
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**Example****Rotating table with concentric circular track**

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Here, the table is moving at a constant angular velocity  $\Omega$ , and the mass is moving on the circular track with speed  $v$  relative to the table.



It is clear that the absolute velocity of point  $A$  is equal to  $\Omega r + v$ , and, as a consequence, the force that the track exerts on  $m$  will be  $\mathbf{R} = -[(\Omega r + v)^2/r] \mathbf{e}_r$ . If, on the other hand, we use expression (6), given the

same observers  $O$  and  $B$  as in the previous example, we have

$$\mathbf{R} - 2m\boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'} - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) = m(\mathbf{a}_{A/B})_{x'y'z'} \quad .$$

Now,  $\boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'} = -\Omega v \mathbf{e}_r$ ,  $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) = -\Omega^2 r \mathbf{e}_r$ , and  $(\mathbf{a}_{A/B})_{x'y'z'} = -(v^2/r) \mathbf{e}_r$ , thus,

$$\mathbf{R} = -(2m\Omega v + \Omega^2 r + \frac{v^2}{r}) \mathbf{e}_r = -\frac{(\Omega r + v)^2}{r} \mathbf{e}_r.$$

The term  $(v^2/r)$  is the acceleration of  $A$  with respect to the table, the term  $\Omega^2 r$  is the acceleration of a point situated at  $A$  and fixed with respect to the table, and, finally, the term  $2\Omega v$  is extra. This final term is the acceleration due to the Coriolis effect. It is present because  $B$  is rotating, and it is required to obtain the “correct” result for acceleration.

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