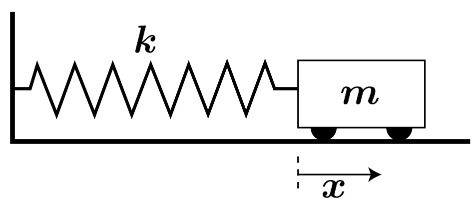
Lecture D31 : Linear Harmonic Oscillator

Spring-Mass System



Spring Force $F = -kx$, $\qquad k > 0$

Newton's Second Law

$$m\ddot{x} + kx = 0$$

(Define) Natural frequency (and period)

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \left(\tau = \frac{2\pi}{\omega_n}\right)$$

Equation of a linear harmonic oscillator

$$\boxed{\ddot{x} + \omega_n^2 x = 0}$$

Solution

General solution

$$x(t) = A\cos\omega_n t + B\sin\omega_n t$$

or,

$$x(t) = C\sin(\omega_n t + \phi)$$

Initial conditions

$$x(0) = x_0 \qquad \dot{x}(0) = \dot{x}_0$$

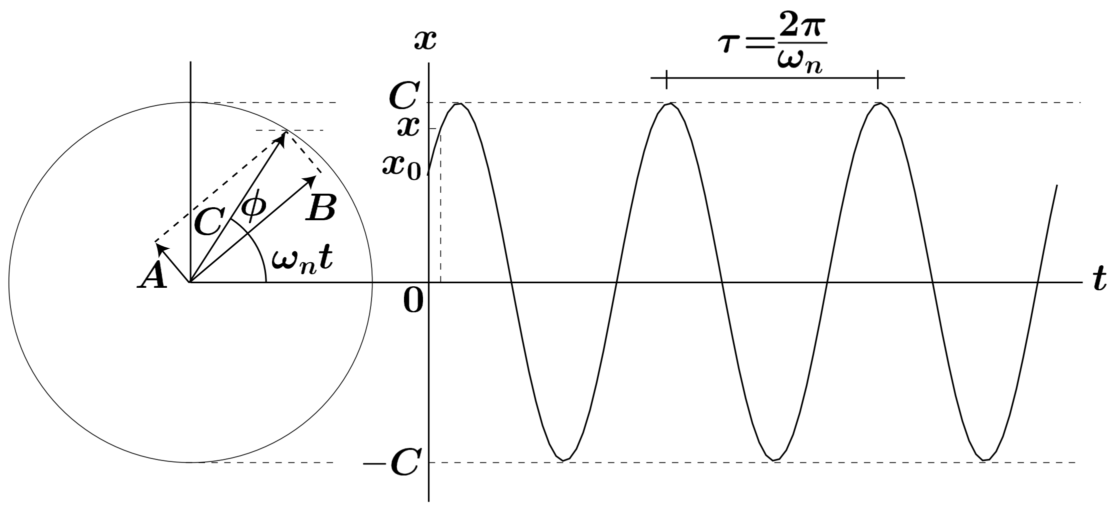
Solution,

$$x(t) = x_0\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t$$

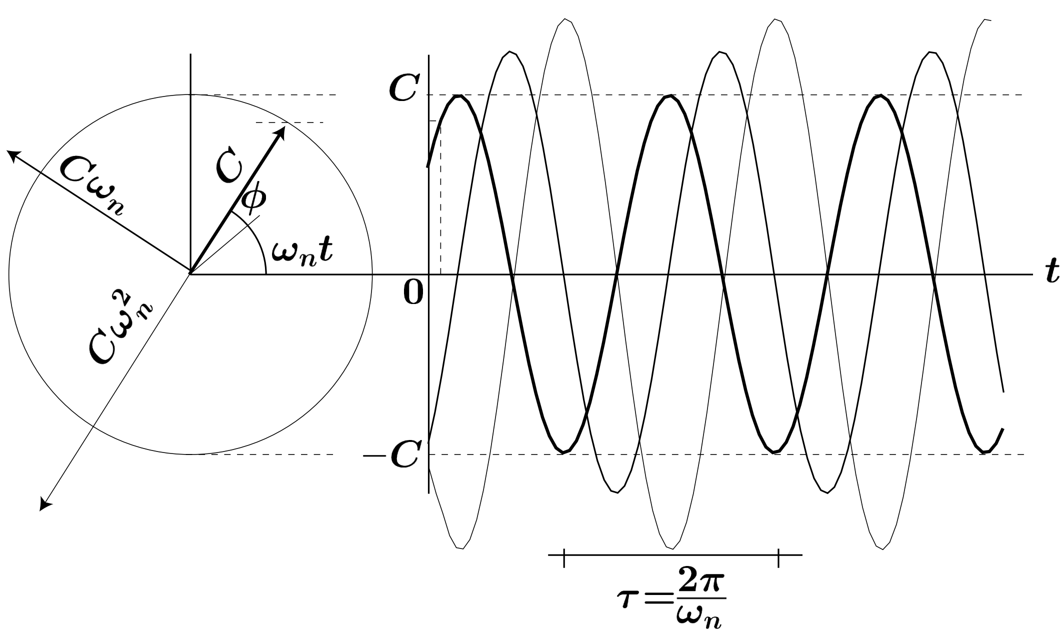
or,

$$x(t) = \sqrt{x_0^2 + (\dot{x}_0/\omega_n)^2}\sin(\omega_n t + \tan^{-1}(\frac{x_0\omega_n}{\dot{x}_0}))$$

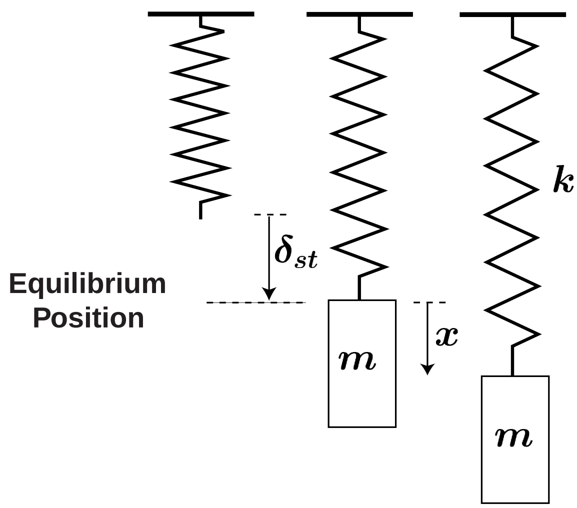
Graphical Representation



Displacement, Velocity and Acceleration



Energy Conservation



No dissipation

$$T + V = \text{constant}$$

Potential Energy

$$V = \frac{1}{2}k(x + \delta_{st})^2 - \frac{1}{2}k\delta_{st}^2 - mgx$$

At Equilibrium $-k\delta_{st} + mg = 0$,

$$V = \frac{1}{2}kx^2$$

Energy Conservation (cont'd)

Kinetic Energy

$$\frac{1}{2}m\dot{x}^2$$

Conservation of energy

$$\frac{d}{dt}(T+V) = m\dot{x}\ddot{x} + kx\dot{x} = 0$$

Governing equation

$$m\ddot{x} + kx = 0$$

Above represents a **very** general way of deriving equations of motion (Lagrangian Mechanics)

Energy Conservation (cont'd)

If $V = 0$ at the equilibrium position,

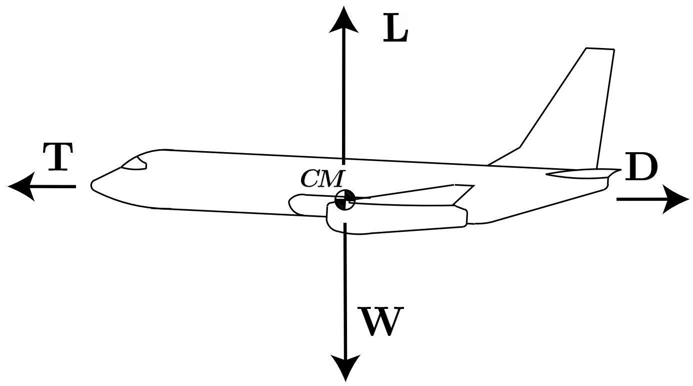
$$
\begin{array}{llll}
V = 0 & T = T_{max} & \text{for} & x = 0 \\
V = V_{max} & T = 0 & \text{for} & x = x_{max}
\end{array}
$$

$$\rightarrow \qquad T_{max} = V_{max}$$

Examples

- Spring-mass systems
- Rotating machinery
- Pendulums (small amplitude)
- Oscillating bodies (small amplitude)
- Aircraft motion (Phugoid)
- Waves (String, Surface, Volume, etc.)
- Circuits
- ...

The Phugoid



Idealized situation :

- Small perturbations (h', v') about *steady level flight* $(h_0,\ v_0)$

$$h = h_0 + h' \qquad v = v_0 + v'$$

- $L = W\ (\equiv mg)$ for $v = v_0$, but $L \sim v^2$,

$$\frac{L}{mg} = \frac{v^2}{v_0^2} \approx \frac{v_0^2}{v_0^2}\ (1 + 2\frac{v'}{v_0} + \ldots)$$

The Phugoid (cont'd)

- Vertical momentum equation

$$m\ddot{h} = L - mg$$

$$\ddot{h} = g(1 + 2\frac{v'}{v_0} - 1)$$

- Energy conservation $T = D$

$$mgh_0 + \frac{1}{2}mv_0^2 = mgh + \frac{1}{2}mv^2$$

$$\text{(to first order)} \rightarrow \quad gh' + v_0 v' = 0$$

- Equations of motion

$$\ddot{h}' + 2\ \frac{g^2}{v_0^2}\ h' \ = \ 0$$

$$\ddot{v}' + 2\ \frac{g^2}{v_0^2}\ v' \ = \ 0$$

The Phugoid (cont'd)

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h' and v' satisfy a *Harmonic Oscillator Equation*

Natural frequency and Period

$$\omega_n = \frac{\sqrt{2}g}{v_0} \qquad \tau = \sqrt{2}\pi\frac{v_0}{g}$$

Light aircraft $v_0 \sim 150$ft/s $\quad \longrightarrow \quad \tau \sim 20$s

Solution

$$\begin{aligned} h' &= A\sin\left(\frac{\sqrt{2}g}{v_0}t\right) \\ v' &= -\frac{g}{v_0}A\sin\left(\frac{\sqrt{2}g}{v_0}t\right) \end{aligned}$$

The Phugoid (cont'd)

Integrate v' equation

$$x' = \frac{A}{\sqrt{2}} \cos\left(\frac{\sqrt{2}g}{v_0} t\right)$$

$$x = x_0 + x'$$