

## Lecture D17 - Conservation Laws for Systems of Particles

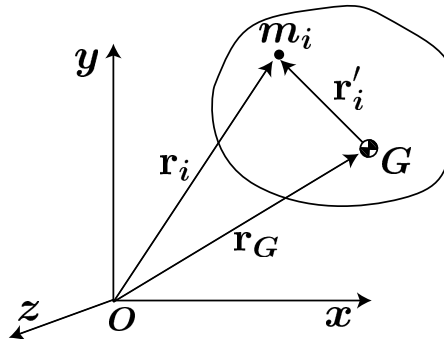
In this lecture, we will revisit the application of Newton's second law to a system of particles and derive some useful relationships expressing the conservation of angular momentum.

### Center of Mass

Consider a system made up of  $n$  particles. A typical particle,  $i$ , has mass  $m_i$ , and, at the instant considered, occupies the position  $\mathbf{r}_i$  relative to a frame  $xyz$ . We can then define the center of mass,  $G$ , as the point whose position vector,  $\mathbf{r}_G$ , is such that,

$$\mathbf{r}_G = \frac{1}{m} \left( \sum_{i=1}^n m_i \mathbf{r}_i \right) . \quad (1)$$

Here,  $m$  is the total mass of the system given by  $m = \sum_{i=1}^n m_i$ .



It is important to note that the center of mass is a property of the system and does not depend on the reference frame used. In particular, if we change the location of the origin  $O$ ,  $\mathbf{r}_G$  will change, but the absolute position of the point  $G$  will not. Often, it will be convenient to describe the motion of particle  $i$  as the motion of  $G$  plus the motion of  $i$  relative to  $G$ . To this end, we introduce the relative position vector,  $\mathbf{r}'_i$ , and write,

$$\mathbf{r}_i = \mathbf{r}_G + \mathbf{r}'_i . \quad (2)$$

It follows immediately, from 1 and 2, that,

$$\sum_{i=1}^n m_i \mathbf{r}'_i = \sum_{i=1}^n m_i (\mathbf{r}_i - \mathbf{r}_G) = \sum_{i=1}^n m_i \mathbf{r}_i - m \mathbf{r}_G = 0 . \quad (3)$$

# Forces

We will consider two types of forces acting on the particles:

*External* forces arising outside the system. We will denote the resultant of all the external forces acting on particle  $i$  as  $\mathbf{F}_i$ .

*Internal* forces due to pairwise particle interactions. Let  $\mathbf{f}_{ij}$  denote the force that particle  $j$  exerts on particle  $i$ . We shall assume that this force is directed along the line joining particle  $j$  and particle  $i$ . This implies that  $(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{f}_{ij} = \mathbf{0}$ . We shall also assume that these internal forces obey Newton's third law. That is,  $\mathbf{f}_{ij} = -\mathbf{f}_{ji}$ , where  $\mathbf{f}_{ji}$  is the force that particle  $i$  exerts on particle  $j$ . The total internal force on particle  $i$  is then,

$$\sum_{j=1, j \neq i}^n \mathbf{f}_{ij},$$

and, if we sum over all particles, we have

$$\sum_{i=1}^n \sum_{j=1, j \neq i}^n \mathbf{f}_{ij} = \mathbf{0},$$

since, for every force,  $\mathbf{f}_{ij}$ , there is an equal and opposite force,  $\mathbf{f}_{ji}$ .

## Conservation of Linear Momentum

The linear momentum of the system is defined as,

$$\mathbf{L} = \sum_{i=1}^n m_i \mathbf{v}_i. \quad (4)$$

From equation 2, we have that  $\mathbf{v}_i = \dot{\mathbf{r}}_i = \dot{\mathbf{r}}_G + \dot{\mathbf{r}}'_i$ , which, combined with the above equation, gives,

$$\mathbf{L} = \sum_{i=1}^n m_i (\dot{\mathbf{r}}_G + \dot{\mathbf{r}}'_i) = \sum_{i=1}^n m_i \mathbf{v}_G + \frac{d}{dt} \left( \sum_{i=1}^n m_i \mathbf{r}'_i \right) = m \mathbf{v}_G, \quad (5)$$

since  $\sum_{i=1}^n m_i \mathbf{r}'_i = \mathbf{0}$ . We now consider the time variation of the linear momentum. If we assume that the reference frame  $xyz$  is inertial, then, starting from equation 4, we have,

$$\dot{\mathbf{L}} = \sum_{i=1}^n m_i \mathbf{a}_i = \sum_{i=1}^n (\mathbf{F}_i + \sum_{j=1, j \neq i}^n \mathbf{f}_{ij}) = \sum_{i=1}^n \mathbf{F}_i = \mathbf{F},$$

where  $\mathbf{F}$  is the sum of all external forces acting on the system. Thus, for a system of particles, we have that,

$$\dot{\mathbf{L}} = \mathbf{F}. \quad (6)$$

Note that, from equation 5, we can also write  $\dot{\mathbf{L}} = m \mathbf{a}_G$ .

# Conservation of Angular Momentum

Since the angular momentum is defined with respect to a point in space, we will consider several cases, using a different reference point for each case.

## Conservation of Angular Momentum about a Fixed Point $O$

The angular momentum of a system of particles about a fixed point,  $O$ , is the sum of the angular momentum of the individual particles,

$$\mathbf{H}_O = \sum_{i=1}^n (\mathbf{r}_i \times m_i \mathbf{v}_i) . \quad (7)$$

The time variation of  $\mathbf{H}_O$  can be written as,

$$\dot{\mathbf{H}}_O = \sum_{i=1}^n (\dot{\mathbf{r}}_i \times m_i \mathbf{v}_i) + \sum_{i=1}^n (\mathbf{r}_i \times m_i \dot{\mathbf{v}}_i) = \mathbf{0} + \sum_{i=1}^n (\mathbf{r}_i \times (\mathbf{F}_i + \sum_{j=1, j \neq i}^n \mathbf{f}_{ij})) = \sum_{i=1}^n (\mathbf{r}_i \times \mathbf{F}_i) .$$

Here, we have used two facts. First, the cross product of two parallel vectors is zero. Second,  $\mathbf{r}_i \times \mathbf{f}_{ij} + \mathbf{r}_j \times \mathbf{f}_{ji} = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{f}_{ij} = \mathbf{0}$ , and, therefore, the internal forces have no net effect on the total angular momentum change of the particle system. Thus, we have that

$$\dot{\mathbf{H}}_O = \mathbf{M}_O , \quad (8)$$

where  $\mathbf{M}_O = \sum_{i=1}^n (\mathbf{r}_i \times \mathbf{F}_i)$  is the total moment, about  $O$ , of the applied external forces. Note that this moment needs to be computed as the sum of the individual moments. In general  $\mathbf{M}_O \neq \mathbf{r}_G \times \mathbf{F}$ , unless the line of action of the the external forces resultant,  $\mathbf{F}$ , passes through  $G$ .

## Conservation of Angular Momentum about $G$

The angular momentum about  $G$  is given by,

$$\mathbf{H}_G = \sum_{i=1}^n (\mathbf{r}'_i \times m_i \mathbf{v}_i) . \quad (9)$$

Taking the time derivative of equation 2, we obtain

$$\mathbf{v}_i = \dot{\mathbf{r}}_i = \dot{\mathbf{r}}_G + \dot{\mathbf{r}}'_i = \mathbf{v}_G + \mathbf{v}'_i . \quad (10)$$

Inserting this expression into equation 9, we obtain

$$\mathbf{H}_G = \sum_{i=1}^n (\mathbf{r}'_i \times m_i (\dot{\mathbf{r}}_G + \dot{\mathbf{r}}'_i)) = \sum_{i=1}^n (\mathbf{r}'_i \times m_i \mathbf{v}'_i) , \quad (11)$$

since  $\sum_{i=1}^n (\mathbf{r}'_i \times m_i \dot{\mathbf{r}}_G) = -\dot{\mathbf{r}}_G \times \sum_{i=1}^n m_i \mathbf{r}'_i = \mathbf{0}$  (see equation 3). We note that equations 9 and 11 give us alternative representations for  $\mathbf{H}_G$ . Equation 9 is called the *absolute* angular momentum (since it involves absolute velocities,  $\mathbf{v}_i$ ), whereas equation 11 is called the *relative* angular momentum (since it involves velocities,  $\mathbf{v}'_i$ , relative to  $G$ ). It turns out that when  $G$  is chosen to be the origin for the relative velocities,

both the absolute and relative angular momentum are identical. As we shall see, this will simplify our analysis significantly. In general, the absolute and relative angular momentum with respect to an arbitrary point are not the same.

We can now go back to equation 9 and consider the time variation of  $\mathbf{H}_G$ ,

$$\dot{\mathbf{H}}_G = \sum_{i=1}^n (\dot{\mathbf{r}}'_i \times m_i(\mathbf{v}_G + \dot{\mathbf{r}}'_i)) + \sum_{i=1}^n (\mathbf{r}'_i \times m_i \dot{\mathbf{v}}_i) = \mathbf{0} + \sum_{i=1}^n (\mathbf{r}'_i \times \mathbf{F}_i). \quad (12)$$

In the above equation, the term  $\dot{\mathbf{r}}'_i \times m_i \dot{\mathbf{r}}'_i$  is clearly zero, and  $\sum_{i=1}^n (\mathbf{r}'_i \times m_i \mathbf{v}_G) = -\mathbf{v}_G \times \sum_{i=1}^n m_i \mathbf{r}'_i = -\mathbf{v}_G \times d(\sum_{i=1}^n m_i \mathbf{r}'_i)/dt = \mathbf{0}$ . Thus, we have that

$$\dot{\mathbf{H}}_G = \mathbf{M}_G. \quad (13)$$

Here,  $\mathbf{M}_G = \sum_{i=1}^n (\mathbf{r}'_i \times \mathbf{F}_i)$ , is the total moment, about  $G$ , of the applied external forces.

The above expression is very powerful and allows us to solve, with great simplicity, a large class of problems in rigid body dynamics. Its power lies in the fact that it is applicable in very general situations:

- In the derivation of equation 13, we have made no assumptions about the motion of the center of mass,  $G$ . That is, equation 13 is valid even when  $G$  is *accelerated*.
- We have implicitly assumed that the reference frame used to describe  $\mathbf{r}'_i$  in equation 10 is non-rotating with respect to the fixed frame  $xyz$  (otherwise, we would have written  $\dot{\mathbf{r}}'_i = \dot{\mathbf{v}}'_i + \boldsymbol{\omega}' \times \mathbf{r}'_i$ , with  $\boldsymbol{\omega}'$ , the angular velocity of the frame considered). It is not difficult to show that equation 13 is still valid if the reference frame rotates, provided the angular velocity is *constant*. If the reference frame rotates with a constant angular velocity, the angular momentum will differ from that of equations, 9 and 11 by a constant, but equation 13 still will be valid (see approach 2 in example below).
- The angular momentum of the particle system about the center of mass,  $G$ , can be evaluated using two alternative representations, 9 and 11. In particular, equation 11 allows us to determine  $\mathbf{H}_G$  without making any reference to the absolute velocities,  $\mathbf{v}_i$ , or position vectors,  $\mathbf{r}_i$ .

Finally, by combining equations 7 and 9, the angular momentum about a fixed point,  $O$ , can be expressed as a function of the angular momentum about the center of mass, as,

$$\mathbf{H}_O = \mathbf{r}_G \times m\mathbf{v}_G + \mathbf{H}_G.$$

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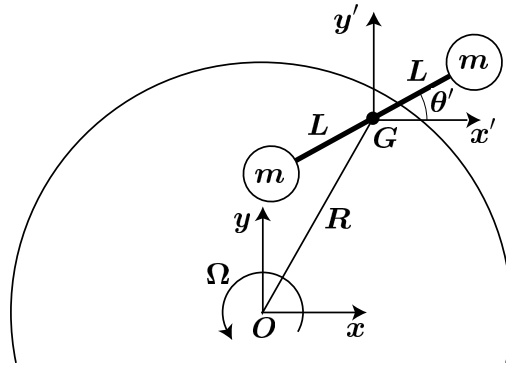
### Example

### Rotating dumbbell [1]

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We consider a steadily spinning turntable with angular velocity  $\boldsymbol{\Omega}$ , with a symmetric dumbbell mounted at a distance  $R$  from the center. The dumbbell is free to spin about its midpoint. We assume that the rod connecting the two masses is massless. Therefore, we can model the dumbbell as a system consisting of two particles. We want to determine the relative motion of the dumbbell. We will consider three different approaches:

1) Use equation 13 with parallel axes

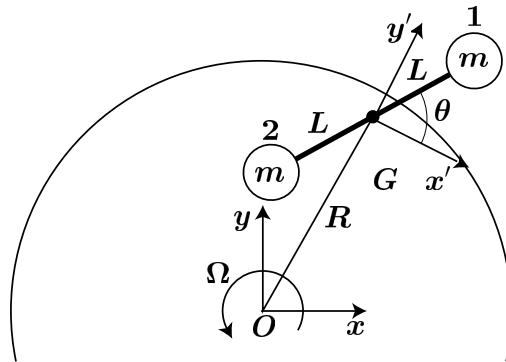


In this case, we are clearly under the assumptions of equation 13, and, therefore, we have that the sum of the moments of all external forces must equal the change in angular momentum. Since the dumbbell is symmetric, the center of mass is at the support point. The only external force to the system (force at the support) has a zero moment with respect to  $G$ , that is,  $\mathbf{M}_G = \mathbf{0}$ . Therefore,  $\dot{\mathbf{H}}_G = 0$ , or  $\mathbf{H}_G = \text{constant}$ . The constant value of  $\mathbf{H}_G$  is determined by the initial conditions. The velocity of the masses relative to  $G$  is  $L\dot{\theta}'$ , and the magnitude of the angular momentum of the system about  $G$  is simply,

$$H_G = 2LmL\dot{\theta}' = 2mL^2\dot{\theta}' .$$

Since  $H_G$  is constant, we have that the angular velocity,  $\dot{\theta}'$ , relative to the parallel axes  $x'y'$ , must also be a constant.

2) Equation 13 with rotating axes



Here, we assume that the local axes  $x'y'$  are rigidly attached to the turntable. Since the angular velocity of the turntable is constant, we can also apply equation 13, and conclude that  $\dot{\theta}$  must be a constant. Note that although  $\theta$  is different from  $\theta'$  (from approach 1),  $\dot{\theta}'$  and  $\dot{\theta}$  only differ by a constant ( $\dot{\theta} = \dot{\theta}' + \Omega$ ), and, hence, this result is compatible with that established in the previous approach (i.e. the integration constants will be different). Note that, in this case, the angular momentum relative

to the rotating  $x'y'$  axes differs from the angular momentum in approach 1 (where the axes remained parallel) by a constant ( $= 2Lm\Omega$ ).

3) Solve directly from Newton's second law

Here, we will calculate the acceleration of each mass with respect to a fixed reference frame. Using the same axes as in approach 2, the acceleration of particle 1,  $\mathbf{a}_1$ , relative to the fixed axes  $xy$ , is

$$\mathbf{a}_1 = \mathbf{a}_G + (\mathbf{a}_1)_{x'y'} + 2\boldsymbol{\Omega} \times (\mathbf{v}_1)_{x'y'} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}'_1) ,$$

Here,  $(\mathbf{v}_1)_{x'y'}$  and  $(\mathbf{a}_1)_{x'y'}$  are, respectively, the velocity and acceleration of particle 1 observed by the rotating frame, and are given by

$$\begin{aligned} \mathbf{v}_1 &= L\dot{\theta}\mathbf{u}_\theta \\ \mathbf{a}_1 &= -L\dot{\theta}^2\mathbf{u}_r + L\ddot{\theta}\mathbf{u}_\theta , \end{aligned}$$

where  $\mathbf{u}_r$  and  $\mathbf{u}_\theta$  are the local polar coordinates unit vectors for particle 1. Also,

$$\mathbf{a}_G = -\Omega^2 R \mathbf{j}' = -\Omega^2 R (\sin \theta \mathbf{u}_r + \cos \theta \mathbf{u}_\theta) ,$$

where  $\mathbf{j}'$  is the unit vector in the direction of  $y'$ . Thus,

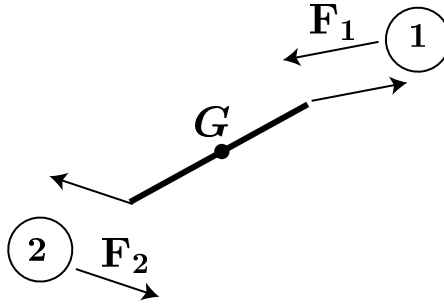
$$\mathbf{a}_1 = (-\Omega^2 R \sin \theta - L\dot{\theta}^2 - 2\Omega L\dot{\theta} - \Omega^2 L)\mathbf{u}_r + (-\Omega^2 R \cos \theta + L\ddot{\theta})\mathbf{u}_\theta .$$

Similarly, for particle 2,

$$\mathbf{a}_2 = (-\Omega^2 R \sin \theta_2 - L\dot{\theta}_2^2 - 2\Omega L\dot{\theta}_2 - \Omega^2 L)\mathbf{u}_{r2} + (-\Omega^2 R \cos \theta_2 + L\ddot{\theta}_2)\mathbf{u}_{\theta 2} .$$

Here,  $\theta_2 = \theta + \pi$  is the circumferential coordinate for mass 2, and  $\mathbf{u}_{r2} (= -\mathbf{u}_r)$  and  $\mathbf{u}_{\theta 2} (= -\mathbf{u}_\theta)$  are the local unit radial and circumferential vectors for mass 2.

Since  $\mathbf{a}_1$  and  $\mathbf{a}_2$  are referred to the fixed frame, we can obtain the forces acting on the masses by applying Newton's second law. Thus,  $\mathbf{F}_1 = m\mathbf{a}_1$  and  $\mathbf{F}_2 = m\mathbf{a}_2$ , where  $\mathbf{F}_1$  and  $\mathbf{F}_2$  denote the forces on particles 1 and 2, respectively. Now, we observe that any force on the masses will be transmitted through the rod.



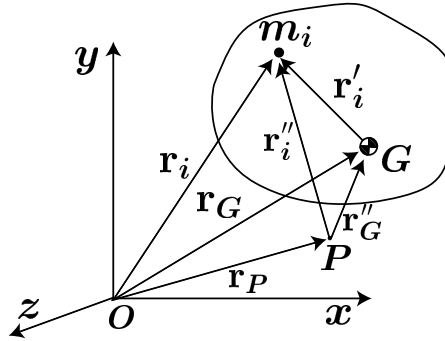
Since we are assuming that the mass of the rod connecting the two masses is zero, the tangential component of the two forces acting on the rod should be in equilibrium (otherwise the rod would experience an infinite angular acceleration). Thus,  $(F_1)_\theta + (F_2)_\theta = 0$ . This results in,

$$m(-\Omega^2 R \cos \theta + L\ddot{\theta}) + m(-\Omega^2 R \cos \theta_2 + L\ddot{\theta}_2) = m(-\Omega^2 R \cos \theta + L\ddot{\theta}) + m(\Omega^2 R \cos \theta + L\ddot{\theta}) = 0 ,$$

which, in turn, implies that  $\ddot{\theta} = 0$ , consistent with the previous results.

## Conservation of Angular Momentum about an arbitrary point $P$

In some situations, it may be convenient to refer the applied moments to an arbitrary point,  $P$ .



The sum of the moments, about point  $P$ ,  $M_P$ , will be given by

$$M_P = M_G + r''_G \times F .$$

Using equations 13 and 6, we have that,

$$\dot{H}_G + r''_G \times m a_G = M_P .$$

It is also possible to derive expressions for the time variation of the *absolute* angular momentum,  $H_P$ , and *relative* angular momentum,  $(H_P)_{rel}$  taken about point  $P$ . Note that, in general,  $H_P \neq (H_P)_{rel}$ ; and that  $\dot{H}_P \neq M_P$  and  $(\dot{H}_P)_{rel} \neq M_P$ . The expressions for the time variation of the angular momentum about an arbitrary point,  $P$ , involve the velocity and acceleration of point  $P$ , and can be found in reference [2].

## Kinetic Energy for Systems of Particles

Here, we derive the expression for the kinetic energy of a system of particles that will be used in the following lectures. A typical particle,  $i$ , will have a mass  $m_i$ , an absolute velocity  $v_i$ , and a kinetic energy  $T_i = (1/2)m_i v_i \cdot v_i = (1/2)m_i v_i^2$ .

The total kinetic energy of the system,  $T$ , is simply the sum of the kinetic energies for each particle,

$$T = \sum_{i=1}^n T_i = \sum_{i=1}^n \frac{1}{2} m_i v_i^2 .$$

It is convenient to decompose the velocity of each particle,  $\mathbf{v}_i$ , into the velocity of the center of mass,  $\mathbf{v}_G$ , and the velocity relative to the center of mass,  $\dot{\mathbf{r}}'_i$ . Then,

$$T = \sum_{i=1}^n \frac{1}{2} m_i (\mathbf{v}_G + \dot{\mathbf{r}}'_i) \cdot (\mathbf{v}_G + \dot{\mathbf{r}}'_i) = \sum_{i=1}^n \frac{1}{2} m_i (v_G^2 + 2\mathbf{v}_G \cdot \dot{\mathbf{r}}'_i + \dot{r}'_i{}^2) = \frac{1}{2} m v_G^2 + \sum_{i=1}^n \frac{1}{2} m_i \dot{r}'_i{}^2 ,$$

since  $\mathbf{v}_G \cdot \sum_{i=1}^n m_i \dot{\mathbf{r}}'_i = 0$ , and  $\sum_{i=1}^n m_i = m$ . Thus, we see that the kinetic energy of a system of particles equals the kinetic energy of a particle of mass  $m$  moving with the velocity of the center of mass, plus the kinetic energy due to the motion of the particles relative to the center of mass,  $G$ .

#### ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition  
4/1, 4/2, 4/3 (kinetic energy expression only), 4/4, 4/5 (momentum only)

## References

- [1] M. Martinez-Sanchez, *Unified Engineering Notes*, Course 95-96.
- [2] J.L. Merriam and L.G. Kraige *Engineering Mechanics, Dynamics*, Fifth Edition, Wiley, 2002.