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16.07 Dynamics
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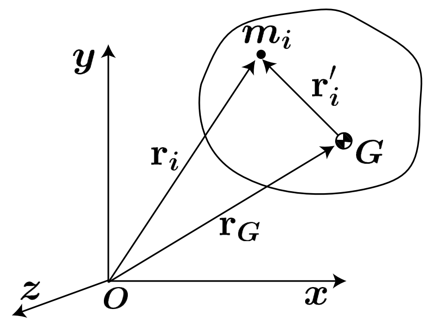
Lecture D19 - 2D Rigid Body Dynamics: Work and Energy

In this lecture, we will revisit the principle of work and energy introduced in lecture D7 for particle dynamics, and extend it to 2D rigid body dynamics.

Kinetic Energy for a 2D Rigid Body

We start by recalling the kinetic energy expression for a system of particles derived in lecture D17,



$$T = \frac{1}{2} m v_G^2 + \sum_{i=1}^{n} \frac{1}{2} m_i \dot{r}_i'^2 \ ,$$

where n is the total number of particles, m_i denotes the mass of particle i, and $\boldsymbol{r}_i'$ is the position vector of particle i with respect to the center of mass, G. Also, $m = \sum_{i=1}^{n} m_i$ is the total mass of the system, and $\boldsymbol{v}_G$ is the velocity of the center of mass. The above expression states that the kinetic energy of a system of particles equals the kinetic energy of a particle of mass m moving with the velocity of the center of mass, plus the kinetic energy due to the motion of the particles relative to the center of mass, G.

For a 2D rigid body, the velocity of all particles relative to the center of mass is a pure rotation. Thus, we can write

$$\dot{\boldsymbol{r}}_i' = \boldsymbol{\omega} \times \boldsymbol{r}_i'.$$

Therefore, we have

$$\sum_{i=1}^{n} \frac{1}{2} m_i \dot{r}_i'^2 = \sum_{i=1}^{n} \frac{1}{2} m_i (\boldsymbol{\omega} \times \boldsymbol{r}_i') \cdot (\boldsymbol{\omega} \times \boldsymbol{r}_i') = \sum_{i=1}^{n} \frac{1}{2} m_i r_i'^2 \omega^2 \ ,$$

where we have used the fact that $\boldsymbol{\omega}$ and $\boldsymbol{r}_i'$ are perpendicular. The term $\sum_{i=1}^{n} m_i r_i'^2$ is easily recognized as the moment of inertia, I_G, about the center of mass, G. Therefore, for a 2D rigid body, the kinetic energy is simply,

$$T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2 \ . \qquad (1)$$

When the body is rotating about a fixed point O, we can write $I_O = I_G + mr_G^2$ and

$$T = \frac{1}{2}mv_G^2 + \frac{1}{2}(I_O - mr_G^2)\omega^2 = \frac{1}{2}I_O\omega^2 \ ,$$

since $v_G = \omega r_G$.

The above expression is also applicable in the more general case when there is no fixed point in the motion, provided that O is replaced by the instantaneous center of rotation. Thus, in general,

$$T = \frac{1}{2}I_C\omega^2 \ .$$

We shall see that, when the instantaneous center of rotation is known, the use of the above expression does simplify the algebra considerably.

Work

Recall that the work done by a force, $\boldsymbol{F}$, over an infinitesimal displacement, $d\boldsymbol{r}$, is $dW = \boldsymbol{F} \cdot d\boldsymbol{r}$. If $\boldsymbol{F}_i^{total}$ denotes the resultant of all forces acting on particle i, then we can write,

$$dW_i = \boldsymbol{F}_i^{total} \cdot d\boldsymbol{r}_i = m_i \frac{d\boldsymbol{v}_i}{dt} \cdot d\boldsymbol{r}_i = m_i \boldsymbol{v}_i \cdot d\boldsymbol{v}_i = d(\frac{1}{2}m_i v_i^2) = d(T_i) \ ,$$

where we have assumed that the velocity is measured relative to an inertial reference frame, and, hence, $\boldsymbol{F}_i^{total} = m_i \boldsymbol{a}_i$. The above equation states that the work done on particle i by the resultant force $\boldsymbol{F}_i^{total}$ is equal to the change in its kinetic energy.

The total work done on particle i, when moving from position 1 to position 2, is

$$(W_i)_{1-2} = \int_1^2 dW_i \ ,$$

and, summing over all particles, we obtain the principle of work and energy for systems of particles,

$$T_1 + \sum_{i=1}^{n} (W_i)_{1-2} = T_2 \ . \qquad (2)$$

The force acting on each particle will be the sum of the *internal forces* caused by the other particles, and the *external forces*. We now consider separately the work done by the internal and external forces.

Internal Forces

We shall assume, once again, that the internal forces due to interactions between particles act along the lines joining the particles, thereby satisfying Newton's third law. Thus, if $\boldsymbol{f}_{ij}$ denotes the force that particle j exerts on particle i, we have that $\boldsymbol{f}_{ij}$ is parallel to $\boldsymbol{r}_i - \boldsymbol{r}_j$, and satisfies $\boldsymbol{f}_{ij} = -\boldsymbol{f}_{ji}$.

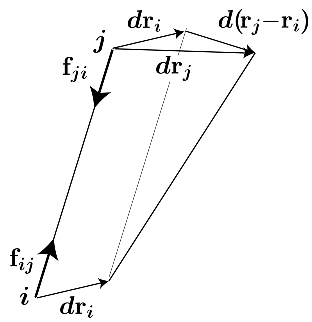
Let us now look at two particles, i and j, undergoing an infinitesimal rigid body motion, and consider the term,

$$\boldsymbol{f}_{ij} \cdot d\boldsymbol{r}_i + \boldsymbol{f}_{ji} \cdot d\boldsymbol{r}_j \ . \qquad (3)$$

If we write $d\boldsymbol{r}_j = d\boldsymbol{r}_i + d(\boldsymbol{r}_j - \boldsymbol{r}_i)$, then,

$$\boldsymbol{f}_{ij} \cdot d\boldsymbol{r}_i + \boldsymbol{f}_{ji} \cdot d\boldsymbol{r}_j = \boldsymbol{f}_{ij} \cdot (d\boldsymbol{r}_i - d\boldsymbol{r}_i) - \boldsymbol{f}_{ij} \cdot d(\boldsymbol{r}_j - \boldsymbol{r}_i) = -\boldsymbol{f}_{ij} \cdot d(\boldsymbol{r}_j - \boldsymbol{r}_i).$$

It turns out that $\boldsymbol{f}_{ij} \cdot d(\boldsymbol{r}_j - \boldsymbol{r}_i)$ is zero, since $d(\boldsymbol{r}_j - \boldsymbol{r}_i)$ is perpendicular to $\boldsymbol{r}_j - \boldsymbol{r}_i$, and hence it is also perpendicular to $\boldsymbol{f}_{ij}$. The orthogonality between $d(\boldsymbol{r}_j - \boldsymbol{r}_i)$ and $\boldsymbol{r}_j - \boldsymbol{r}_i$ follows from the fact that the distance between any two particles in a rigid body must remain constant (i.e. $(\boldsymbol{r}_j - \boldsymbol{r}_i) \cdot (\boldsymbol{r}_j - \boldsymbol{r}_i) = (r_j - r_i)^2 =$ const; thus differentiating, we have $2d(\boldsymbol{r}_j - \boldsymbol{r}_i) \cdot (\boldsymbol{r}_j - \boldsymbol{r}_i) = 0$).



We conclude that, since all the work done by the internal forces can be written as a sum of terms of the form 3, then the contribution of all the internal forces to the term $\sum_{i=1}^{n}(W_i)_{1-2}$ in equation 2, is zero.

External Forces

We have established in the previous section that only the external forces to the rigid body are capable of doing any work. Thus, the total work done on the body will be

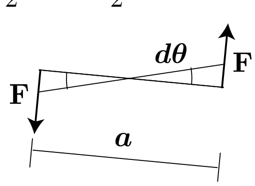
$$\sum_{i=1}^{n}(W_i)_{1-2} = \sum_{i=1}^{n} \int_{(\boldsymbol{r}_i)_1}^{(\boldsymbol{r}_i)_2} \boldsymbol{F}_i \cdot d\boldsymbol{r} \quad ,$$

where $\boldsymbol{F}_i$ is the sum of all the external forces acting on particle i.

Work done by couples

If the sum of the external forces acting on the rigid body is zero, it is still possible to have non-zero work. Consider, for instance, a moment $M = Fa$ acting on a rigid body. If the body undergoes a pure translation, it is clear that all the points in the body experience the same displacement, and, hence, the total work done by a couple is zero. On the other hand, if the body experiences a rotation $d\theta$, then the work done by the couple is

$$dW = F\frac{a}{2}d\theta + F\frac{a}{2}d\theta = F\,ad\theta = Md\theta \ .$$



If M is constant, the work is simply $W_{1-2} = M(\theta^2 - \theta^1)$

Conservative Forces

When the forces can be derived from a potential energy function, V, we say the forces are conservative. In such cases, we have that $\boldsymbol{F} = -\boldsymbol{\nabla} V$, and the work and energy relation in equation 2 takes a particularly simple form. Recall that a necessary, but not sufficient, condition for a force to be conservative is that it must be a function of position only, i.e. $\boldsymbol{F}(\boldsymbol{r})$ and $V(\boldsymbol{r})$. Common examples of conservative forces are gravity (a constant force independent of the height), gravitational attraction between two bodies (a force inversely proportional to the squared distance between the bodies), and the force of a perfectly elastic spring.

The work done by a conservative force between position $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ is

$$W_{1-2} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r} = [-V]_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} = V(\boldsymbol{r}_1) - V(\boldsymbol{r}_2) = V_1 - V_2 \quad .$$

Thus, if we call W_{1-2}^{NC} the work done by all the external forces which are *non conservative*, we can write the general expression,

$$T_1 + V_1 + W_{1-2}^{NC} = T_2 + V_2 \quad .$$

Of course, if all the forces that do work are conservative, we obtain conservation of total energy, which can be expressed as,

$$T + V = \text{constant} \ .$$

Gravity Potential for a Rigid Body

In this case, the potential V_i associated with particle i is simply $V_i = m_i g z_i$, where z_i is the height of particle i above some reference height. The force acting on particle i will then be $\boldsymbol{F}_i = -\boldsymbol{\nabla} V_i$. The work done on the whole body will be

$$\sum_{i=1}^{n} \int_{\boldsymbol{r}_i^1}^{\boldsymbol{r}_i^2} \boldsymbol{f}_i \cdot d\boldsymbol{r}_i = \sum_{i=1}^{n} ((V_i)_1 - (V_i)_2) = \sum_{i=1}^{n} m_i g((z_i)_1 - (z_i)_2 = V_1 - V_2 \quad ,$$

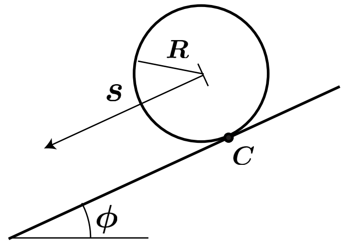
where the gravity potential for the rigid body is simply,

$$V = \sum_{i=1}^{n} m_i g z_i = m g z_G \ ,$$

where z_G is the z coordinate of the center of mass.

Example — ***Cylinder on a Ramp***

We consider a homogeneous cylinder released from rest at the top of a ramp of angle ϕ, and use conservation of energy to derive an expression for the velocity of the cylinder.



Conservation of energy implies that $T+V = T_{initial}+V_{initial}$. Initially, the kinetic energy is zero, $T_{initial} = 0$. Thus, for a later time, the kinetic energy is given by

$$T = V_{initial} - V = mgs\sin\phi \ ,$$

where s is the distance traveled down the ramp. The kinetic energy is simply $T = \frac{1}{2}I_C\omega^2$, where $I_C = I_G + mR^2$ is the moment of inertia about the instantaneous center of rotation C, and ω is the angular velocity. Thus, $I_C\omega^2 = 2mgs\sin\phi$, or,

$$v^2 = \frac{2gs\sin\phi}{1+(I_G/mR^2)} \ ,$$

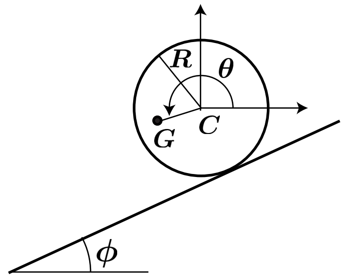
since $\omega = v/R$.

Equilibrium and Stability

If all the forces acting on the body are conservative, then the potential energy can be used very effectively to determine the equilibrium positions of a system and the nature of the stability at these positions. Let us assume that all the forces acting on the system can be derived from a potential energy function, V. It is clear that if $\boldsymbol{F} = -\boldsymbol{\nabla}V = \boldsymbol{0}$ for some position, this will be a point of equilibrium in the sense that if the body is at rest (kinetic energy zero), then there will be no forces (and hence, no acceleration) to change the equilibrium, since the resultant force $\boldsymbol{F}$ is zero. Once equilibrium has been established, the stability of the equilibrium point can be determine by examining the *shape* of the potential function. If the potential function has a *minimum* at the equilibrium point, then the equilibrium will be *stable*. This means that if the potential energy is at a minimum, there is no potential energy left that can be *traded* for kinetic energy. Analogously, if the potential energy is at a *maximum*, then the equilibrium point is *unstable*.

Example ***Equilibrium and Stability***

A cylinder of radius R, for which the center of gravity, G, is at a distance d from the geometric center, C, lies on a rough plane inclined at an angle ϕ.



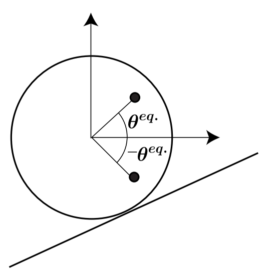
Since gravity is the only external force acting on the cylinder that is capable of doing any work, we can examine the equilibrium and stability of the system by considering the potential energy function. We have $z_C = z_{C0} - R\theta \sin\phi$, where z_{C0} is the value of z_C when $\theta = 0$. Thus, since $d = |CG|$, we have,

$$V = mgz_G = mg(z_C + d\sin\theta) = mg(z_{C0} - R\theta\sin\phi + d\sin\theta) \quad .$$

The equilibrium points are given by $\boldsymbol{\nabla} V = \mathbf{0}$, but, in this case, since the position of the system is uniquely determined by a single coordinate, e.g. θ, we can write

$$\boldsymbol{\nabla} V = \frac{dV}{d\theta}\boldsymbol{\nabla}\theta \ ,$$

which implies that, for equilibrium, $dV/d\theta = mg(-R\sin\phi + d\cos\theta) = 0$, or, $\cos\theta = (R\sin\phi)/d$. If $d < R\sin\phi$, there will be *no equilibrium* positions. On the other hand, if $d \geq R\sin\phi$, then $\theta^{eq.} = \cos^{-1}[(R\sin\phi)/d]$ is an equilibrium point. We note that if $\theta^{eq.}$ is an equilibrium point, then $-\theta^{eq.}$ is also an equilibrium point (i.e. $\cos\theta = \cos(-\theta)$).



In order to study the stability of the equilibrium points, we need to determine whether the potential energy is a maximum or a minimum at these points. Since $d^2V/d\theta^2 = -mgd\sin\theta$, we have that when $\theta^{eq.} < 0$, then $d^2V/d\theta^2 > 0$ and the potential energy is a minimum at that point. Consequently, for $\theta^{eq.} < 0$, the equilibrium is stable. On the other hand, for $\theta^{eq.} > 0$, the equilibrium point is unstable.

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

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