Lectures D25-D26 :
3D Rigid Body Dynamics

12 November 2004

Outline

- Review of Equations of Motion
- Rotational Motion
- Equations of Motion in Rotating Coordinates
- Euler Equations
- Example: Stability of Torque Free Motion
- Gyroscopic Motion
  - Euler Angles
  - Steady Precession
- Steady Precession with $\mathbf{M} = \mathbf{0}$

Equations of Motion

Conservation of **Linear** Momentum

$$\dot{\mathbf{L}} = \mathbf{F}, \qquad \mathbf{L} = m\mathbf{v}_G$$

Conservation of **Angular** Momentum

$$\dot{\mathbf{H}}_G = \mathbf{M}_G, \qquad \mathbf{H}_G = \boldsymbol{I}_G\boldsymbol{\omega}$$

or

$$\dot{\mathbf{H}}_O = \mathbf{M}_O, \qquad \mathbf{H}_O = \boldsymbol{I}_O\boldsymbol{\omega}$$

Equations of Motion in Rotating Coordinates

Angular Momentum

$$\mathbf{H}_G = \boldsymbol{I}_G \boldsymbol{\omega} \qquad\qquad \text{(or } \mathbf{H}_O = \boldsymbol{I}_O \boldsymbol{\omega}\text{)}$$

Time variation

- Non-rotating axes XYZ ($\boldsymbol{I}$ changes)

$$\dot{\mathbf{H}} = \dot{\boldsymbol{I}}\boldsymbol{\omega} + \boldsymbol{I}\dot{\boldsymbol{\omega}} \qquad \ldots\ \dot{\boldsymbol{I}} \text{ big problem!}$$

- Rotating axes xyz ($\boldsymbol{I}$ constant)

$$\dot{\mathbf{H}} = (\dot{\mathbf{H}})_{xyz} + \boldsymbol{\Omega} \times \mathbf{H}$$

$$= \boldsymbol{I}(\dot{\boldsymbol{\omega}})_{xyz} + \boldsymbol{\Omega} \times \mathbf{H}$$

Equations of Motion in Rotating Coordinates

$$\boxed{(\dot{\mathrm{H}})_{xyz} + \mathbf{\Omega} \times \mathrm{H} = \mathrm{M}}$$

or,

$$\dot{H}_x - H_y\Omega_z + H_z\Omega_y = M_x$$

$$\dot{H}_y - H_z\Omega_x + H_x\Omega_z = M_y$$

$$\dot{H}_z - H_x\Omega_y + H_y\Omega_x = M_z$$

xyz axis can be **any** right-handed set of axis, but

. . . will **choose** xyz (Ω) to **simplify** analysis (e.g. I constant)

Example: Parallel Plane Motion

$$\omega_x = \omega_y = 0, \qquad \omega_z \neq 0$$

$$H_x = -I_{xz}\omega_z, \quad H_y = -I_{yz}\omega_z, \quad H_z = I_z\omega_z$$

Body fixed axis $\Omega = \omega$ (and $z \equiv Z$)

$$-I_{xz}\dot{\omega}_z + I_{yz}\omega_z^2 = M_x \qquad (1)$$

$$-I_{yz}\dot{\omega}_z - I_{xz}\omega_z^2 = M_y \qquad (2)$$

$$I_z\dot{\omega}_z = M_z \qquad (3)$$

Solve (3) for ω_z, and then, (1) and (2) for M_x and M_y.

Euler’s Equations

If xyz are principal axes of inertia

- $H_x = I_x\omega_x,\; H_y = I_y\omega_y, H_z = I_z\omega_z$
- $\Omega = \omega$

$$I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z = M_x$$

$$I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x = M_y$$

$$I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y = M_z$$

Euler's Equations

- Body fixed principal axes
- Right-handed coordinate frame
- Origin at:
  - Center of mass G (possibly accelerated)
  - Fixed point O
- Non-linear equations . . . hard to solve
- Solution gives angular velocity components . . . in unknown directions (need to *integrate* $\boldsymbol{\omega}$ to determine orientation).

Example: Stability of Torque Free Motion

Body spinning about principal axis of inertia,

$$\omega_z = \omega, \qquad \omega_x = \omega_y = 0$$

Consider small perturbation

$$\omega_x,\ \omega_y, \ll \omega$$

After initial perturbation $\mathrm{M} = 0$

$$I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z = 0 \qquad (1)$$

$$I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x = 0 \qquad (2)$$

$$I_z\dot{\omega}_z - (I_x - I_y)\underbrace{\omega_x\omega_y}_{\text{small}} = 0 \qquad (3)$$

Example: Stability of Torque Free Motion

From $(3) \;\longrightarrow\; \omega_z \approx \omega \equiv$ constant

Differentiate (1) and substitute value of $\dot{\omega}_y$ from (2),

$\longrightarrow \; \ldots$

$$I_x\ddot{\omega}_x - \frac{(I_y - I_z)(I_z - I_x)}{I_y}\omega^2\omega_x = 0$$

or,

$$\ddot{\omega}_x - \mathcal{A}\omega_x = 0, \qquad \mathcal{A} = -\frac{(I_z - I_y)(I_z - I_x)}{I_x I_y}\omega^2$$

Solutions,

$$\omega_x = Ae^{\sqrt{\mathcal{A}}t} + Be^{-\sqrt{\mathcal{A}}t}$$

Example: Stability of Torque Free Motion

- $\mathcal{A} > 0 \rightarrow \sqrt{\mathcal{A}} > 0 \rightarrow$ Growth → **Unstable**

  $I_x < I_z < I_y,$ or $I_y < I_z < I_x$

- $\mathcal{A} < 0 \rightarrow \sqrt{\mathcal{A}} = i\alpha \rightarrow$ Oscillatory → **Stable**

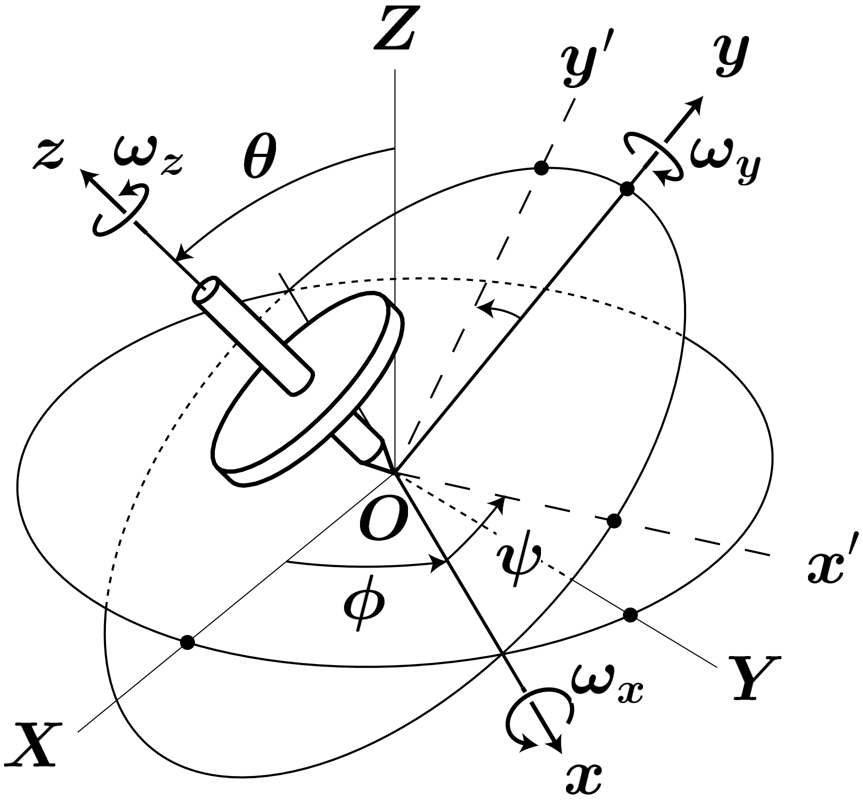
  $I_z > I_x, I_y,$ or $I_z < I_x, I_y$

Gyroscopic Motion

- Bodies symmetric w.r.t.(spin) axis

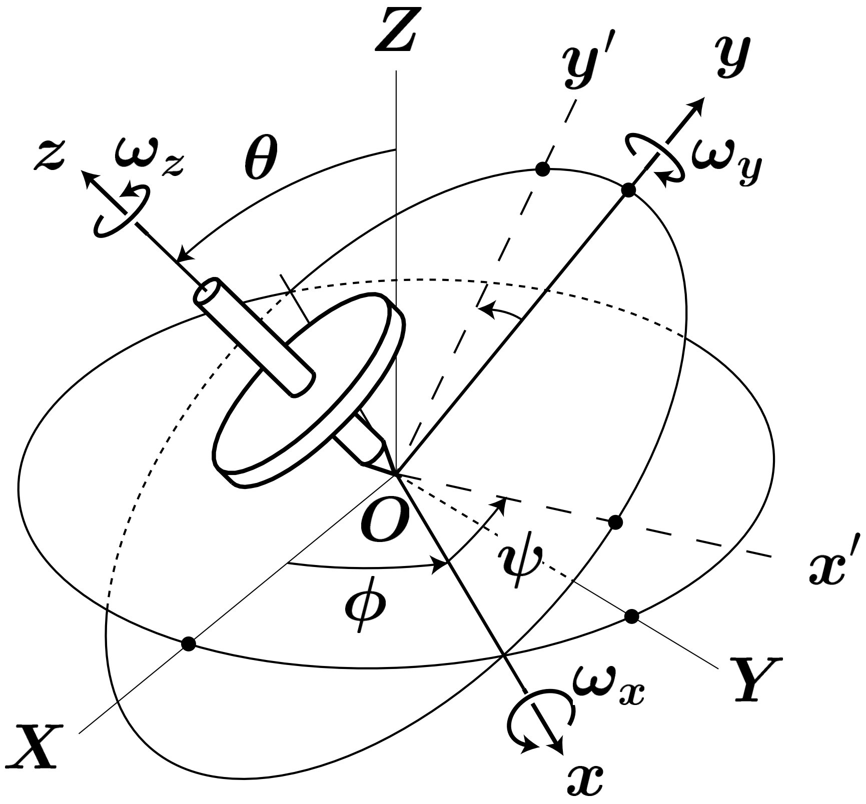
$$I_z = I, \quad I_x = I_y = I_0$$

- Origin at fixed point O (or at G)



Gyroscopic Motion

- XYZ fixed axes
- $x'y'z$ body axes — angular velocity $\boldsymbol{\omega}$
- xyz "working" axes — angular velocity $\boldsymbol{\Omega}$



Gyroscopic Motion Euler Angles

ϕ : **Precession**

θ : **Nutation**

ψ : **Spin**

– position of xyz requires ϕ and $\boldsymbol{\theta}$

– position of $x'y'z$ requires ϕ, $\boldsymbol{\theta}$ and ψ

Relation between $(\boldsymbol{\phi}, \boldsymbol{\theta}, \boldsymbol{\psi})$ and $\boldsymbol{\omega}$,(and $\boldsymbol{\Omega}$)

$$\begin{aligned} \boldsymbol{\omega_x} &= \dot{\boldsymbol{\theta}} & \qquad \boldsymbol{\Omega_x} &= \dot{\boldsymbol{\theta}} \\ \boldsymbol{\omega_y} &= \dot{\boldsymbol{\phi}} \sin \boldsymbol{\theta} & \qquad \boldsymbol{\Omega_y} &= \dot{\boldsymbol{\phi}} \sin \boldsymbol{\theta} \\ \boldsymbol{\omega_z} &= \dot{\boldsymbol{\phi}} \cos \boldsymbol{\theta} + \dot{\boldsymbol{\psi}} & \qquad \boldsymbol{\Omega_z} &= \dot{\boldsymbol{\phi}} \cos \boldsymbol{\theta} \end{aligned}$$

Gyroscopic Motion Euler Angles

Angular Momentum

$$H_x = I_x\omega_x = I_0\dot{\theta}$$

$$H_y = I_y\omega_y = I_0\dot{\phi}\sin\theta$$

$$H_z = I_z\omega_z = I(\dot{\phi}\cos\theta + \dot{\psi})$$

Equation of Motion,

$$\dot{H}_x - H_y\Omega_z + H_z\Omega_y = M_x$$

$$\dot{H}_y - H_z\Omega_x + H_x\Omega_z = M_y$$

$$\dot{H}_z - H_x\Omega_y + H_y\Omega_x = M_z$$

Gyroscopic Motion

Euler Angles

become,

$$I_0(\ddot\theta - \dot\phi^2 \sin\theta\cos\theta) + I\dot\phi\sin\theta(\dot\phi\cos\theta + \dot\psi) = M_x$$

$$I_0(\ddot\phi\sin\theta + 2\dot\phi\dot\theta\cos\theta) - I\dot\theta(\dot\phi\cos\theta + \dot\psi) = M_y$$

$$I(\ddot\psi + \ddot\phi\cos\theta - \dot\phi\dot\theta\sin\theta) = M_z$$

. . . not easy to solve!!

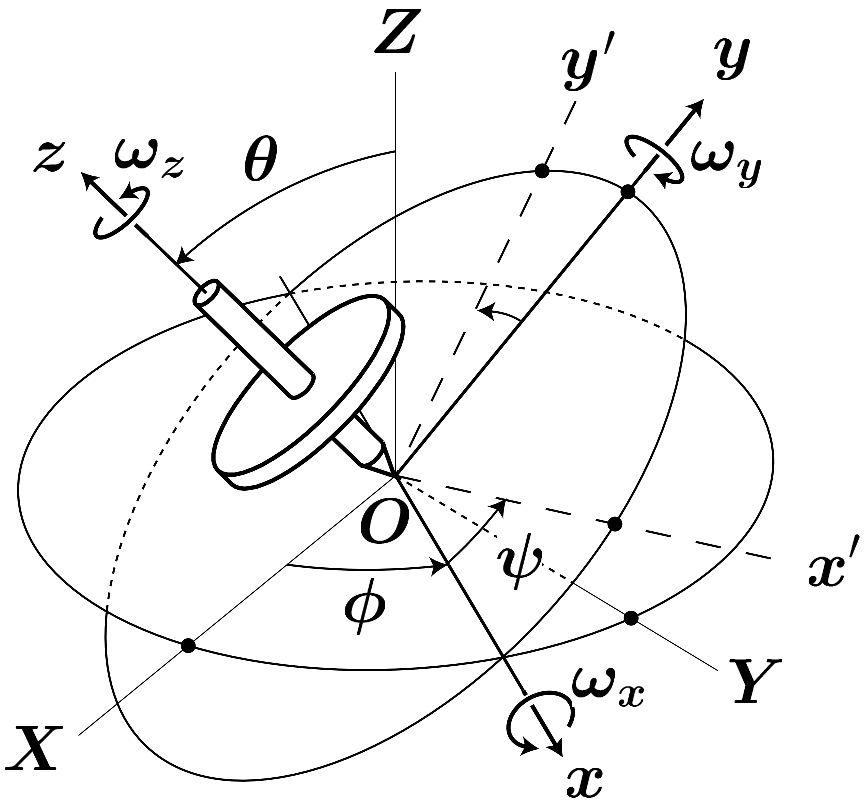
Gyroscopic Motion

Steady Precession

$$\dot{\phi} = \text{constant}, \qquad \ddot{\phi} = 0$$

$$\theta = \text{constant}, \qquad \dot{\theta} = \ddot{\theta} = 0$$

$$\dot{\psi} = \text{constant}, \qquad \ddot{\psi} = 0$$



Gyroscopic Motion Steady Precession

$$\dot{\phi}\sin\theta[I(\dot{\phi}\cos\theta+\dot{\psi})-I_0\dot{\phi}\cos\theta]=M_x$$

$$0=M_y$$

$$0=M_z$$

Also, note that $(\dot{\mathrm{H}})_{xyz}=0\longrightarrow$

H does not change in xyz axes

External Moment

$$M_x=mgz_G\sin\theta$$

Gyroscopic Motion Steady Precession

Then,

$$I\dot{\phi}\dot{\psi} - (I_0 - I)\dot{\phi}^2 \cos\theta = mgz_G$$

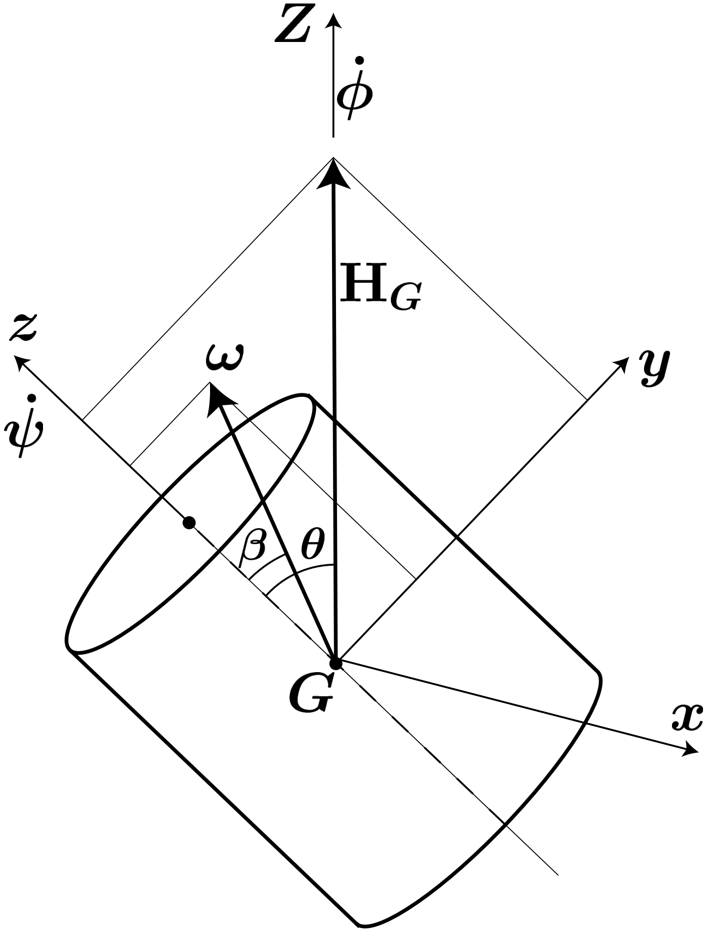
If $\dot{\psi} \gg \dot{\phi}$, or, $\theta \approx \frac{\pi}{2}$,

$$\dot{\phi} = \frac{mgz_G}{I\dot{\psi}}$$

$\dot{\phi}$: precession velocity,
$\dot{\psi}$: spin velocity

Steady Precession with $\mathrm{M} = 0$

$$\mathrm{M}_G = 0 \;\; \Rightarrow \;\; \mathrm{H}_G = \text{constant}$$



$$H_{Gx} = I_0 \omega_x = 0$$

$$H_{Gy} = I_0 \omega_y = H_G \sin\theta$$

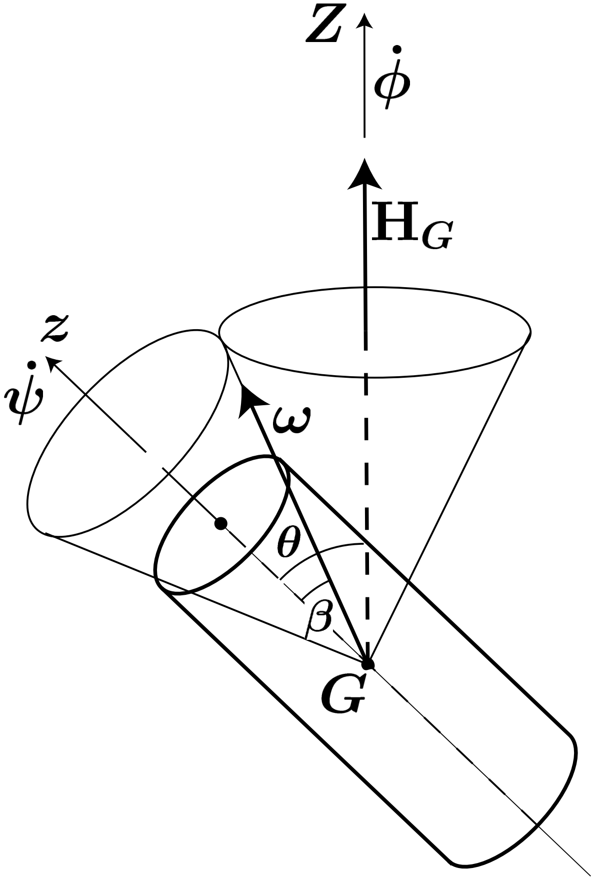
$$H_{Gz} = I \omega_z = H_G \cos\theta$$

$$\tan\theta = \frac{H_{Gy}}{H_{Gz}} = \frac{I_0}{I}\frac{\omega_y}{\omega_z} = \frac{I_0}{I}\tan\beta$$

Steady Precession with $\mathrm{M} = 0$
Direct Precession

From x-component of angular momentum equation,

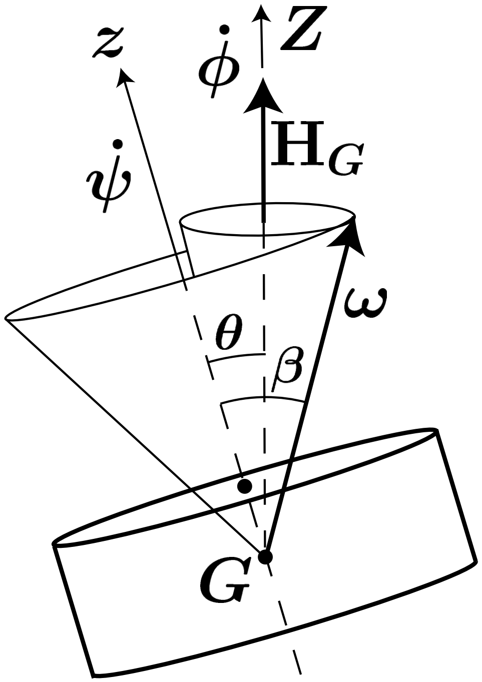
$$\dot{\phi} = \frac{I\dot{\psi}}{(I_0 - I)\cos\theta}$$



If $I_0 > I$, then $\beta < \theta$ ($\tan\theta = (I_0/I)\tan\beta$), $\rightarrow \dot{\phi}$ same sign as $\dot{\psi}$

Steady Precession with $\mathrm{M} = 0$
Retrograde Precession

$$\dot{\phi} = \frac{I\dot{\psi}}{(I_0 - I)\cos\theta}$$



If $I_0 < I$, then $\beta > \theta$ ($\tan\theta = (I_0/I)\tan\beta$), $\rightarrow \dot{\phi}$ and$\dot{\psi}$ have opposite signs