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16.07 Dynamics
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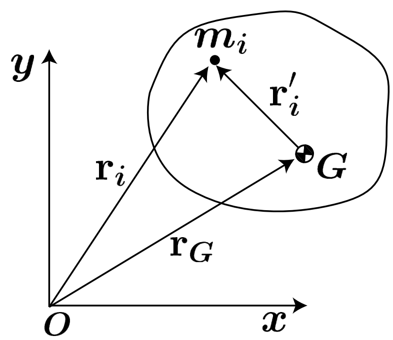
Lecture D18 - 2D Rigid Body Dynamics: Equations of Motion

In this lecture, we will particularize the conservation principles presented in the previous lecture to the case in which the system of particles considered is a 2D rigid body.

Mass Moment of Inertia

In the previous lecture, we established that the angular momentum of a system of particles relative to the center of mass, G, was

$$\boldsymbol{H}_G = \sum_{i=1}^{n} (\boldsymbol{r}_i' \times m_i \boldsymbol{v}_i')$$



When considering a 2D rigid body, the velocity of any point relative to G consists of a pure rotation and, therefore, can be expressed as

$$\boldsymbol{v}_i' = \boldsymbol{\omega} \times \boldsymbol{r}_i' \ ,$$

where $\boldsymbol{\omega}$ is the angular velocity vector perpendicular to the plane of motion. These two equations can be combined to give,

$$\boldsymbol{H}_G = \sum_{i=1}^{n} (\boldsymbol{r}_i' \times m_i(\boldsymbol{\omega} \times \boldsymbol{r}_i')) = \sum_{i=1}^{n} m_i r_i'^2 \boldsymbol{\omega} \ .$$

Here, we have used the vector identity, $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}$, and imposed the fact that $\boldsymbol{r}_i'$ and $\boldsymbol{\omega}$ are perpendicular for 2D planar bodies.

If we define the *mass moment of inertia* of the body, I_G, as

$$I_G = \sum_{i=1}^{n} m_i r_i'^2 \ ,$$

then,

$$\boldsymbol{H}_G = I_G \boldsymbol{\omega} \ .$$

The moment of inertia, I_G, is a scalar quantity. It is a property of the solid which indicates the way in which the mass of the solid is distributed relative to the center of mass. For example, if most of the mass is far away from the center of mass, r'_i will be large, resulting in a large moment of inertia. The dimensions of the moment of inertia are $[M][L^2]$.

When the system of particles is a continuum, the summation over the number of particles is written as an integral,

$$I_G = \int_m r'^2 \, dm = \int_V \rho r'^2 \, dV \ , \tag{1}$$

where ρ is the mass density and V is the volume. In this case, the position of the center of mass, $\boldsymbol{r}_G$, is given by

$$\boldsymbol{r}_G = \frac{\int_m \boldsymbol{r} \, dm}{m} \ ,$$

with $m = \int_m dm$, the total mass of the body.

When considering three dimensional bodies undergoing two dimensional motion, the moment of inertia needs to be defined with respect to an axis perpendicular to the plane of motion. In this case, we can still use equation 1, with r' replaced by the distance to the axis.

It follows from the above definition that the moment of inertia of a composite body about a given point can always be calculated as the sum of the moments of inertia of the different components.

Exercise

Show that:

- the moment of inertia of a homogenous, slender bar of length L and mass M, about an axis perpendicular to the bar and passing through the center of mass, is $I_G = ML^2/12$,
- the moment of inertia of a homogeneous circular ring of mass M and radius R, about an axis perpendicular to the plane of the ring and passing through the center of mass, is $I_G = MR^2$,
- the moment of inertia of a homogeneous circular disc of mass M and radius R, about an axis perpendicular to the plane of the disc and passing through the center of mass, is $I_G = MR^2/2$.

Parallel Axis (Steiner) Theorem

We will often need to find the moment of inertia with respect to a point other than the center of mass. For instance, the moment of inertia with respect to a given point, O, is defined as

$$I_O = \int_m r^2 \, dm = \int_V \rho r^2 \, dV \ .$$

Assuming that O is a fixed point, $\boldsymbol{H}_O = I_O\boldsymbol{\omega}$. If we know I_G, then the moment of inertia with respect to another point, say point O, can be computed easily using the parallel axis theorem. Given the relations $r^2 = \boldsymbol{r}\cdot\boldsymbol{r}$ and $\boldsymbol{r} = \boldsymbol{r}_G + \boldsymbol{r}'$., we can then write,

$$I_O = \int_m r^2\,dm = \int_m (r_G^2 + 2\boldsymbol{r}_G\cdot\boldsymbol{r}' + r'^2)\,dm = r_G^2\int_m dm + 2\boldsymbol{r}_G\cdot\int_m \boldsymbol{r}'\,dm + \int_m r'^2\,dm = mr_G^2 + I_G\ ,$$

since $\int_m \boldsymbol{r}'\,dm = \boldsymbol{0}$. From this expression, it follows that the moment of inertia with respect to an arbitrary point is minimum when the point coincides with G. Hence, the minimum value for the moment of inertia is I_G.

Radius of Gyration

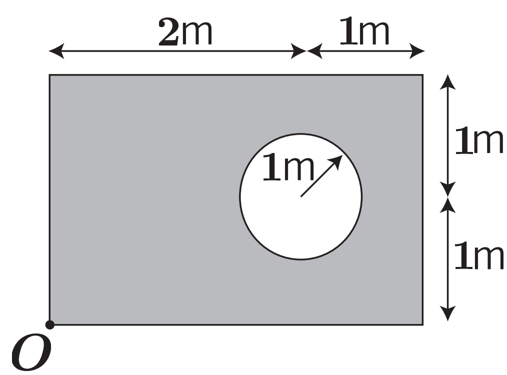
It is common to report the moment of inertia of a rigid body in terms of the radius of gyration, k. This is defined as

$$k = \sqrt{\frac{I}{m}}\ ,$$

and can be interpreted as the root-mean-square of the mass element distances from the axis of rotation.

Example ***Moment of inertia of a rectangular plate with circular cut-out***

We want to determine the moment of inertia of a rectangular plate of mass M kg about an axis perpendicular to the plate and passing through point O.



The density of the plate per unit area is

$$\rho = \frac{M}{A_p - A_c} = 0.3498M\ \text{kg/m}^2\ ,$$

where $A_p = 6\text{m}^2$ is the area of the plate without the cut-out, and $A_c = 3.1416\text{m}^2$ is the area of the circle. The moment of inertia of the plate about its center of mass is $(I_p)_G = \rho A_p(3^2+2^2)/12$. Using the parallel axis theorem, the moment of inertia of the plate about point O will be $(I_p)_O = (I_p)_G + \rho A_p(1.5^2+1^2)$. Similarly, for the circle, $(I_c)_G = \rho A_c 1^2$ and $(I_c)_O = (I_c)_G + \rho A_c(2^2+1^2)$. Finally, the moment of inertia of the plate with the circle cut-out about point O will be,

$$I_O = (I_p)_O - (I_c)_O = 4.333\rho A_p - 6\rho A_c = 2.5012M\ \text{kg m}^2\ .$$

The radius of gyration is

$$k_O = \sqrt{I_O/M} = 1.5815\text{m} \ .$$

See Reference [1] (Appendix B) for a more examples on how to calculate moments of inertia.

Equations of Motion

The equations describing the general motion of a rigid body follow from the conservation laws for systems of particles established in the last lecture. Since the general motion of a 2D rigid body can be determined by three parameters (e.g. x and y position of the center of mass, and rotation angle), we will need to supply three equations. Conservation of linear momentum yields one vector equation, or two scalar equations. The additional condition is conservation of angular momentum. We saw in the last lecture that there are several ways to express conservation of angular momentum. In principle, they are all equivalent, but, depending on the problem situation, the use of a particular form may greatly simplify the problem.

The conservation of linear momentum for a system of particles yields the vector equation,

$$m\boldsymbol{a}_G = \boldsymbol{F} \ , \tag{2}$$

where m is the body mass, $\boldsymbol{a}_G$ is the acceleration of the center of mass, and $\boldsymbol{F}$ is the sum of the external forces acting on the body.

Conservation of angular momentum about the center of mass, G, is simply

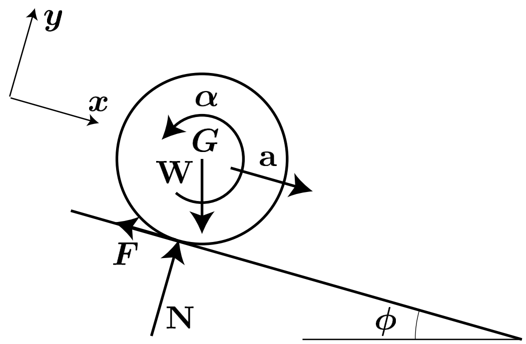
$$I_G\boldsymbol{\alpha} = \boldsymbol{M}_G \ , \tag{3}$$

where I_G is the moment of inertia about the center of mass, $\boldsymbol{\alpha}$ is the angular acceleration (recall that the angular acceleration (and angular velocity), is the same for all points in a rigid body), and $\boldsymbol{M}_G$ is the total moment of the applied external forces (and moments) about the center of mass.

Although 3 is a vector equation, $\boldsymbol{\alpha}$ and $\boldsymbol{M}_G$ are always perpendicular to the plane of motion, and, therefore, equation 3 only yields one scalar equation. It is interesting to note the similarity between equations 2 and 3. The moment of inertia, I_G, can be interpreted as a measure of the body's resistance to changing its angular velocity as a result of applied external moments.

If the applied forces and moments on the body are known, equations 2 and 3 can be integrated to determine the body's trajectory. In most practical problems, however, one does not know a priori all the forces but a combination of some forces and some characteristics of the motion. In such cases where forces are unknown, the additional equations must come from kinematic conditions (e.g. if the cylinder rolls without sliding, the acceleration of the center of mass is related to the angular velocity acceleration; also, the direction of the acceleration of the center of mass will be known in this case).

Let us consider a uniform cylinder of mass m and radius R rolling without slipping down a ramp of angle ϕ.



We consider the fixed reference frame, xy, shown in the picture. The equations of motion, 2 and 3, are, in this case,

$$\begin{aligned} m\ddot{x}_G &= W\sin\phi - F \\ m\ddot{y}_G &= N - W\cos\phi \\ I_G\alpha &= -FR \end{aligned}$$

In these equations, $W = mg$, but the normal force, N, and the friction force, F, are unknown. These two additional unknowns can be determined if we provide two additional kinematic conditions. First, we have that $\ddot{y}_G = 0$, from which we can determine N as

$$N = W\cos\phi \; .$$

Second, since the cylinder rolls without sliding, we have that $\ddot{x}_G = -R\alpha$. Solving for $\ddot{x}_G$, we obtain

$$\ddot{x}_G = \frac{g\sin\phi}{1 + (I_G/mR^2)} \; , \tag{4}$$

and $F = (I_G\ddot{x}_G)/R^2$. For the uniform cylinder, we have that $I_G = mR^2/2$ and $\ddot{x}_G = (2g\sin\phi)/3$. If there was no friction, F would be zero, and the acceleration would be simply $\ddot{x}_G = g\sin\phi$. Also, if instead of having a uniform disc, we had a uniform ring with all the mass concentrated at the rim, then $I_G = mR^2$ and $\ddot{x}_G = (g\sin\phi)/2$.

Rotation about a Fixed Axis

For cases in which there is a fixed point in the body, the motion of the body can be described with a single parameter (e.g. the rotation angle). In principle, we could still consider equations 2 and 3 and use the kinematic conditions to enforce that the motion of the fixed point is zero. Alternatively, the analysis is often simplified if we consider the conservation of angular momentum about the fixed point directly. In this case, we have,

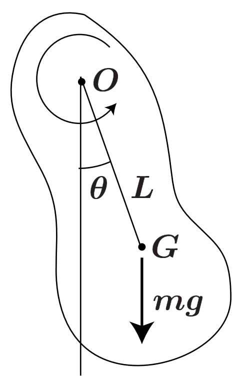
$$I_O \boldsymbol{\alpha} = \boldsymbol{M}_O \ . \tag{5}$$

where I_O is the moment of inertia about the fixed point, O, $\boldsymbol{\alpha}$ is the angular acceleration, and $\boldsymbol{M}_O$ is the sum of external moments about O.

A common mistake often made when solving problems in rigid body dynamics is to apply equation 5 where O is taken as the instantaneous center of rotation. In general, this will lead to erroneous results. Although the instantaneous center of rotation does indeed have zero velocity, it does not in general have zero acceleration. Therefore, in such cases, expression 5 is not applicable. It can be verified that if in the previous example we use this strategy, we will obtain the right answer provided the center of mass of the cylinder is at the geometric center. If the mass distribution were non-uniform, this approach would fail, whereas the correct application of equations 2 and 3 would not (Try it!).

Example ***Compound Pendulum***

An example of a rigid body rotating about a fixed axis is the compound pendulum. A compound pendulum is a rigid body hinged, without friction, about a horizontal axis offset from its center of mass, and acted upon by its own weight as an external force.



The conservation of angular momentum about O implies that

$$I_O \ddot{\theta} = M_O = -mgL \sin\theta \ ,$$

or,

$$\ddot{\theta} = -\frac{g}{I_O/mL} \sin\theta \ .$$

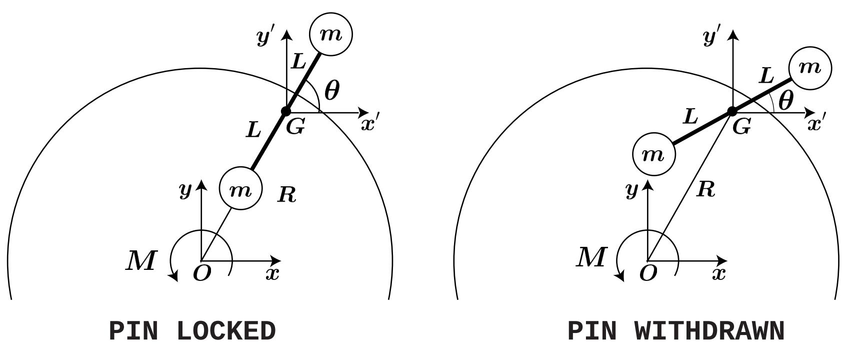
Comparing it with the equation for a simple pendulum, we see that the motion of a compound pendulum is identical to the motion of a simple pendulum of *equivalent* length, L_{equiv},

$$L_{equiv} = \frac{I_O}{mL} \ .$$

Once the angular acceleration and velocity, of the pendulum has been determined, we can use the conservation of linear momentum (written in $r-\theta$ coordinates) and determine the reaction forces at the hinge. Try it!

Example ***Dumbbell on a Rotating Table***

Here, we consider a dumbbell mounted on turntable at a distance R from the center. There is a pin that can be used to lock the dumbbell into the position shown in the diagram. When the pin is withdrawn, the dumbbell is free to rotate about its midpoint. We shall assume that the moment of inertia of the turntable with respect to O is I_t, and that the rod connecting the two masses has negligible mass. The system is initially at rest. We want to determine the angular acceleration of the turntable when an external torque, M, is applied, for the cases where a) the locking pin is in place, and b) the pin is withdrawn.



The absolute angular momentum of the system with respect to the fixed point O will be the sum of the angular momentum of the turntable, $(\boldsymbol{H}_O)_t$, plus the angular momentum of the dumbbell, $(\boldsymbol{H}_O)_d$. The magnitude of $(\boldsymbol{H}_O)_t$ is simply $I_t\omega_t$. The angular momentum of the dumbbell with respect to O is

$$\boldsymbol{H}_O = \sum_{i=1}^{2} \boldsymbol{r}_i \times m\boldsymbol{v}_i = \sum_{i=1}^{2} (\boldsymbol{r}_G + \boldsymbol{r}'_i) \times m\boldsymbol{v}_i = 2m\boldsymbol{r}_G \times \boldsymbol{v}_G + \boldsymbol{H}_G \ .$$

Here, $\boldsymbol{H}_G$ is the absolute angular momentum of the dumbbell with respect to its center of mass, G (which is equal to the relative angular momentum). If we use parallel axis theorem, then,

$$\boldsymbol{H}_G = \sum_{i=1}^{2} \boldsymbol{r}'_i \times m(\boldsymbol{\omega}_d \times \boldsymbol{r}'_i) \ ,$$

where $\boldsymbol{\omega}_d$ is the angular velocity of the dumbbell. Therefore,

$$H_O = I_t\omega_t + 2mR^2\omega_t + 2mL^2\omega_d$$

and,

$$M = \dot{H}_O = (I_t + 2mR^2)\dot{\omega}_t + 2mL^2\dot{\omega}_d \ .$$

Now, we can consider the two different situations. When the pin is in place, the turntable plus the dumbbell behave like a single rigid body. Therefore, $\omega_d = \omega_t$, and the acceleration of the whole system is,

$$\dot{\omega}_t = \frac{M}{I_t + 2m(R^2 + L^2)} \ .$$

On the other hand, when the pin is withdrawn, we can write $(\boldsymbol{M}_G)_d = (\dot{\boldsymbol{H}}_G)_d$ for the dumbbell in isolation. Since there are no applied moments and the initial velocity is zero, $(\boldsymbol{H}_G)_d = 0$. As a result, $\omega_d = 0$, and

$$\dot{\omega}_t = \frac{M}{I_t + 2mR^2} \ .$$

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

6/1, 6/2, 6/3, 6/4, 6/5

References

[1] J.L. Merriam and L.G. Kraige *Engineeering Mechanics, Dynamics*, Fifth Edition, Wiley, 2002.