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16.07 Dynamics
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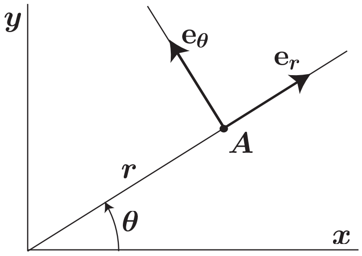
Lecture D5 - Other Coordinates Systems

In this lecture we will look at some other common systems of coordinates. We will present polar coordinates in two dimensions and cylindrical and spherical coordinates in three dimensions. We shall see that these systems are particularly useful for certain classes of problems. Like in the case of intrinsic coordinates presented in the previous lecture, the reference frame changes from point to point. However, for the coordinate systems to be presented below, the reference frame depends *only* on the position of the particle. This is in contrast with the intrinsic coordinates, where the reference frame is a function of the position, as well as the path.

Polar Coordinates $(r - \theta)$

In polar coordinates, the position of a particle A, is determined by the value of the radial distance to the origin, r, and the angle that the radial line makes with an arbitrary fixed line, such as the x axis. Thus, the trajectory of a particle will be determined if we know r and θ as a function of t, i.e. $r(t), \theta(t)$. The directions of increasing r and θ are defined by the orthogonal unit vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$.

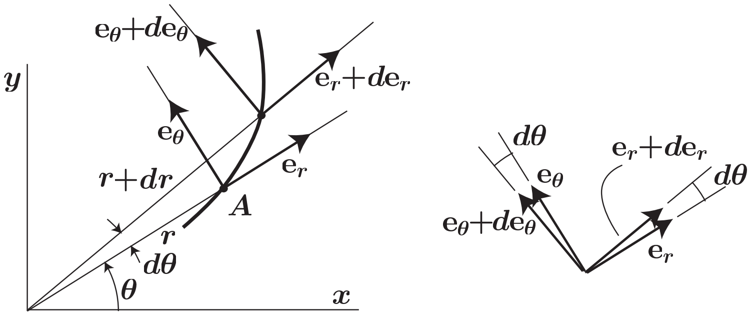


The position vector of a particle has a magnitude equal to the radial distance, and a direction determined by $\boldsymbol{e}_r$. Thus,

$$\boldsymbol{r} = r\boldsymbol{e}_r \ . \tag{1}$$

Since the vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are clearly different from point to point, their variation will have to be considered when calculating the velocity and acceleration.

Over an infinitesimal interval of time dt, the coordinates of point A will change from (r, θ), to $(r + dr,\ \theta + d\theta)$ as shown in the diagram.



We note that the vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ do not change when the coordinate r changes. Thus, $d\boldsymbol{e}_r/dr = \mathbf{0}$ and $d\boldsymbol{e}_\theta/dr = \mathbf{0}$. On the other hand, when θ changes to $\theta + d\theta$, the vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are rotated by an angle $d\theta$. From the diagram, we see that $d\boldsymbol{e}_r = d\theta \boldsymbol{e}_\theta$, and that $d\boldsymbol{e}_\theta = -d\theta \boldsymbol{e}_r$. This is because their magnitudes in the limit are equal to the unit vector as radius times $d\theta$ in radians. Dividing through by $d\theta$, we have,

$$\frac{d\boldsymbol{e}_r}{d\theta} = \boldsymbol{e}_\theta, \quad \text{and} \quad \frac{d\boldsymbol{e}_\theta}{d\theta} = -\boldsymbol{e}_r \ .$$

Multiplying these expressions by $d\theta/dt \equiv \dot{\theta}$, we obtain,

$$\frac{d\boldsymbol{e}_r}{d\theta}\frac{d\theta}{dt} \equiv \frac{d\boldsymbol{e}_r}{dt} = \dot{\theta}\boldsymbol{e}_\theta, \quad \text{and} \quad \frac{d\boldsymbol{e}_\theta}{dt} = -\dot{\theta}\boldsymbol{e}_r \ . \tag{2}$$

Note ***Alternative calculation of the unit vector derivatives***

An alternative, more mathematical, approach to obtaining the derivatives of the unit vectors is to express $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ in terms of their cartesian components along $\boldsymbol{i}$ and $\boldsymbol{j}$. We have that

$$\begin{aligned} \boldsymbol{e}_r &= \cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j} \\ \boldsymbol{e}_\theta &= -\sin\theta \boldsymbol{i} + \cos\theta \boldsymbol{j} \ . \end{aligned}$$

Therefore, when we differentiate we obtain,

$$\begin{aligned} \frac{d\boldsymbol{e}_r}{dr} = 0, \qquad \frac{d\boldsymbol{e}_r}{d\theta} &= -\sin\theta \boldsymbol{i} + \cos\theta \boldsymbol{j} \equiv \boldsymbol{e}_\theta \\ \frac{d\boldsymbol{e}_\theta}{dr} = 0, \qquad \frac{d\boldsymbol{e}_\theta}{d\theta} &= -\cos\theta \boldsymbol{i} - \sin\theta \boldsymbol{j} \equiv -\boldsymbol{e}_r \ . \end{aligned}$$

Velocity vector

We can now derive expression (1) with respect to time and write

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{r}\,\boldsymbol{e}_r + r\,\dot{\boldsymbol{e}}_r \ ,$$

or, using expression (2), we have

$$\boldsymbol{v} = \dot{r}\,\boldsymbol{e}_r + r\dot{\theta}\,\boldsymbol{e}_\theta \ . \tag{3}$$

Here, $v_r = \dot{r}$ is the *radial velocity* component, and $v_\theta = r\dot{\theta}$ is the *circumferential velocity* component. We also have that $v = \sqrt{v_r^2 + v_\theta^2}$. The radial component is the rate at which $\boldsymbol{r}$ changes magnitude, or stretches, and the circumferential component, is the rate at which $\boldsymbol{r}$ changes direction, or swings.

Acceleration vector

Differentiating again with respect to time, we obtain the acceleration

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{r}\,\boldsymbol{e}_r + \dot{r}\,\dot{\boldsymbol{e}}_r + \dot{r}\dot{\theta}\,\boldsymbol{e}_\theta + r\ddot{\theta}\,\boldsymbol{e}_\theta + r\dot{\theta}\,\dot{\boldsymbol{e}}_\theta$$

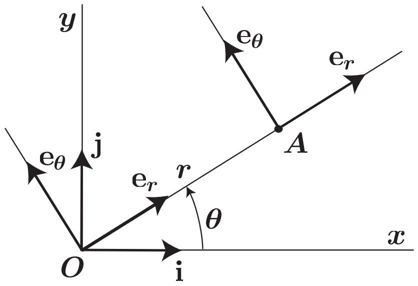
Using the expressions (2), we obtain,

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\,\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\boldsymbol{e}_\theta \ , \tag{4}$$

where $a_r = (\ddot{r} - r\dot{\theta}^2)$ is the *radial acceleration* component, and $a_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta})$ is the *circumferential acceleration* component. Also, we have that $a = \sqrt{a_r^2 + a_\theta^2}$.

Change of basis

In many practical situations, it will be necessary to transform the vectors expressed in polar coordinates to cartesian coordinates and vice versa.

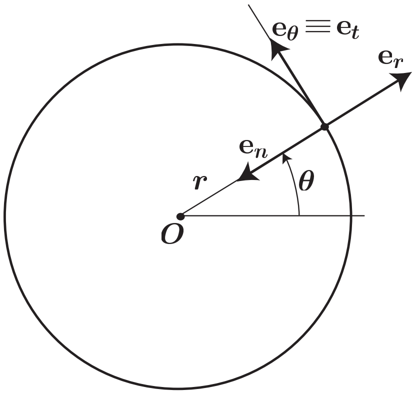


Since we are dealing with free vectors, we can translate the polar reference frame for a given point (r, θ), to the origin, and apply a standard change of basis procedure. This will give, for a generic vector $\boldsymbol{A}$,

$$\begin{pmatrix} A_r \\ A_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \end{pmatrix} \ .$$

Example ***Circular motion***

Consider as an illustration, the motion of a particle in a circular trajectory having angular velocity $\omega = \dot{\theta}$, and angular acceleration $\alpha = \dot{\omega}$. We see that, for this problem, the circumferential and radial directions are very similar to the intrinsic tangential and normal directions. The only difference is that in polar coordinates, the radial direction points outwards, whereas, in intrinsic coordinates, the normal direction always points towards the center of curvature O.



In polar coordinates, the equation of the trajectory is

$$r = R = \text{constant}, \qquad \theta = \omega t + \frac{1}{2}\alpha t^2 .$$

The velocity components are

$$v_r = \dot{r} = 0, \qquad \text{and} \qquad v_\theta = r\dot{\theta} = R(\omega + \alpha t) = v \ ,$$

and the acceleration components are,

$$a_r = \ddot{r} - r\dot{\theta}^2 = -R(\omega + \alpha t)^2 = -\frac{v^2}{R}, \qquad \text{and} \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = R\alpha = a_t \ ,$$

where we clearly see that, $a_r \equiv -a_n$, and that $a_\theta \equiv a_t$.
In cartesian coordinates, we have for the trajectory,

$$x = R\cos(\omega t + \frac{1}{2}\alpha t^2), \quad x = R\sin(\omega t + \frac{1}{2}\alpha t^2) \ .$$

For the velocity,

$$v_x = -R(\omega + \alpha t)\sin(\omega t + \frac{1}{2}\alpha t^2), \quad v_y = R(\omega + \alpha t)\cos(\omega t + \frac{1}{2}\alpha t^2) \ ,$$

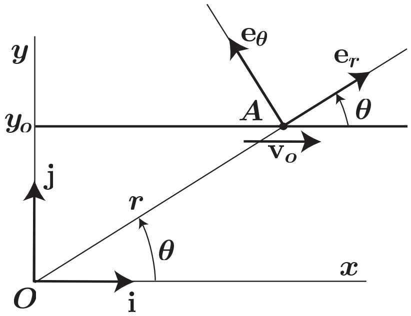
and, for the acceleration,

$$a_x = -R(\omega{+}\alpha t)^2\cos(\omega t{+}\frac{1}{2}\alpha t^2){-}R\alpha\sin(\omega t{+}\frac{1}{2}\alpha t^2), \quad a_y = -R(\omega{+}\alpha t)^2\sin(\omega t{+}\frac{1}{2}\alpha t^2){+}R\alpha\cos(\omega t{+}\frac{1}{2}\alpha t^2) \ .$$

We observe that, for this problem, the result is much simpler when expressed in polar (or intrinsic) coordinates.

Example ***Motion on a straight line***

Here we consider the problem of a particle moving with constant velocity v_0, along a horizontal line $y = y_0$.



Assuming that at $t = 0$ the particle is at $x = 0$, the trajectory and velocity components in cartesian coordinates are simply,

$$x = v_0 t \qquad y = y_0$$

$$v_x = v_0 \qquad v_y = 0$$

$$a_x = 0 \qquad a_y = 0 \quad .$$

Note that, for this problem, cartesian and intrinsic coordinates are virtually identical (for a straight line the normal direction is not defined, in which case we can arbitrarily choose any direction perpendicular to the tangent direction). In polar coordinates, on the other hand, we have,

$$r = \sqrt{v_0^2 t^2 + y_0^2} \qquad \theta = \tan^{-1}(\frac{y_0}{v_0 t})$$

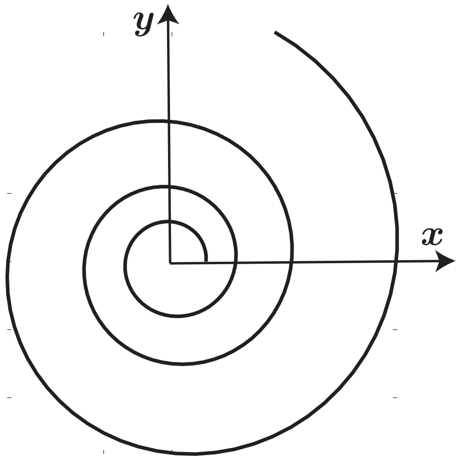
$$v_r = \dot{r} = v_0 \cos\theta \qquad v_\theta = r\dot{\theta} = -v_0 \sin\theta$$

$$a_r = \ddot{r} - r\dot{\theta}^2 = 0 \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad .$$

Here, we see that the expressions obtained in cartesian coordinates are simpler than those obtained using polar coordinates.

Example ***Spiral motion (Kelppner/Kolenkow)***

A particle moves with $\dot{\theta} = \omega =$ constant and $r = r_0 e^{\beta t}$, where r_0 and β are constants.



We shall show that for certain values of β, the particle moves with $a_r = 0$.

$$\begin{aligned} \boldsymbol{a} &= (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta \\ &= (\beta^2 r_0 e^{\beta t} - r_0 e^{\beta t}\omega^2)\boldsymbol{e}_r + 2\beta r_0 \omega e^{\beta t}\boldsymbol{e}_\theta \end{aligned}$$

If $\beta = \pm\omega$, the radial part of $\boldsymbol{a}$ vanishes. It seems quite surprising that when $r = r_0 e^{\beta t}$, the particle moves with zero radial acceleration. The error is in thinking that $\ddot{r}$ makes the only contribution to a_r; the term $-r\dot{\theta}^2$ is also part of the radial acceleration, and cannot be neglected.

The paradox is that even though $a_r = 0$, the radial velocity $v_r = \dot{r} = r_0 \beta e^{\beta t}$ is increasing rapidly in time. In polar coordinates

$$v_r \neq \int a_r(t)dt \ ,$$

because this integral does not take into account the fact that $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are functions of time.

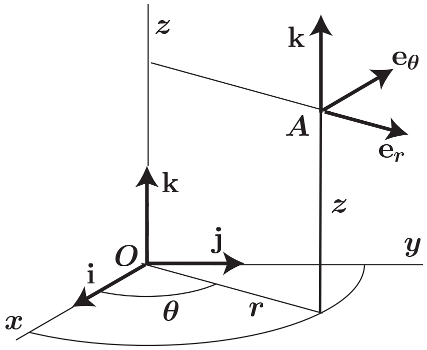
Equations of Motion

In two dimensional polar $r\theta$ coordinates, the force and acceleration vectors are $\boldsymbol{F} = F_r \boldsymbol{e}_r + F_\theta \boldsymbol{e}_\theta$ and $\boldsymbol{a} = a_r \boldsymbol{e}_r + a_\theta \boldsymbol{e}_\theta$. Thus, in component form, we have,

$$\begin{aligned} F_r &= m\, a_r = m\,(\ddot{r} - r\dot{\theta}^2) \\ F_\theta &= m\, a_\theta = m\,(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \ . \end{aligned}$$

Cylindrical Coordinates $(r - \theta - z)$

Polar coordinates can be extended to three dimensions in a very straightforward manner. We simply add the z coordinate, which is then treated in a cartesian like manner. Every point in space is determined by the r and θ coordinates of its projection in the xy plane, and its z coordinate.



The unit vectors $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ and $\boldsymbol{k}$, expressed in cartesian coordinates, are,

$$\begin{aligned} \boldsymbol{e}_r &= \cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j} \\ \boldsymbol{e}_\theta &= -\sin\theta \boldsymbol{i} + \cos\theta \boldsymbol{j} \ , \end{aligned}$$

and their derivatives,

$$\dot{\boldsymbol{e}}_r = \dot{\theta}\boldsymbol{e}_\theta, \quad \dot{\boldsymbol{e}}_\theta = -\dot{\theta}\boldsymbol{e}_r, \quad \dot{\boldsymbol{k}} = \mathbf{0} \ .$$

The kinematic vectors can now be expressed relative to the unit vectors $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ and $\boldsymbol{k}$. Thus, the position vector is

$$\boldsymbol{r} = r\,\boldsymbol{e}_r + z\,\boldsymbol{k} \ ,$$

and the velocity,

$$\boldsymbol{v} = \dot{r}\,\boldsymbol{e}_r + r\dot{\theta}\,\boldsymbol{e}_\theta + \dot{z}\,\boldsymbol{k} \ ,$$

where $v_r = \dot{r}$, $v_\theta = r\dot{\theta}$, $v_z = \dot{z}$, and $v = \sqrt{v_r^2 + v_\theta^2 + v_z^2}$. Finally, the acceleration becomes

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\,\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\boldsymbol{e}_\theta + \ddot{z}\boldsymbol{k} \ ,$$

where $a_r = \ddot{r} - r\dot{\theta}^2$, $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$, $a_z = \ddot{z}$, and $a = \sqrt{a_r^2 + a_\theta^2 + a_z^2}$.
Note that when using cylindrical coordinates, r is not the modulus of $\boldsymbol{r}$. This is somewhat confusing, but it is consistent with the notation used by most books. Whenever we use cylindrical coordinates, we will write $|\boldsymbol{r}|$ explicitly, to indicate the modulus of $\boldsymbol{r}$, i.e. $|\boldsymbol{r}| = \sqrt{r^2 + z^2}$.

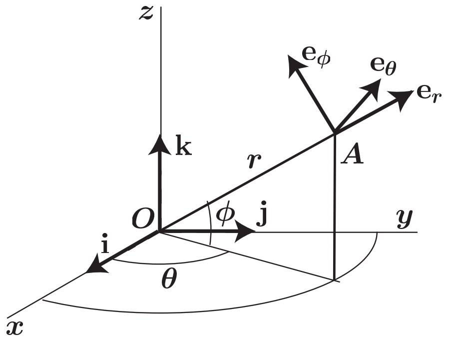
Equations of Motion

In cylindrical $r\theta z$ coordinates, the force and acceleration vectors are $\boldsymbol{F} = F_r\boldsymbol{e}_r + F_\theta\boldsymbol{e}_\theta + F_z\boldsymbol{e}_z$ and $\boldsymbol{a} = a_r\boldsymbol{e}_r + a_\theta\boldsymbol{e}_\theta + a_z\boldsymbol{e}_z$. Thus, in component form we have,

$$\begin{aligned} F_r &= m\,a_r = m\,(\ddot{r} - r\dot{\theta}^2) \\ F_\theta &= m\,a_\theta = m\,(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \\ F_z &= m\,a_z = m\,\ddot{z} \ . \end{aligned}$$

Spherical Coordinates $(r - \theta - \phi)$

In spherical coordinates, we utilize two angles and a distance to specify the position of a particle, as in the case of radar measurements, for example.



The unit vectors written in cartesian coordinates are,

$$\begin{aligned}
\boldsymbol{e}_r &= \cos\theta\,\cos\phi\,\boldsymbol{i} + \sin\theta\,\cos\phi\,\boldsymbol{j} + \sin\phi\,\boldsymbol{k}\\
\boldsymbol{e}_\theta &= -\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j}\\
\boldsymbol{e}_\phi &= -\cos\theta\,\sin\phi\,\boldsymbol{i} - \sin\theta\,\sin\phi\,\boldsymbol{j} + \cos\phi\,\boldsymbol{k}
\end{aligned}$$

The derivation of expressions for the velocity and acceleration follow easily once the derivatives of the unit vectors are known. In three dimensions, the geometry is somewhat more involved, but the ideas are the same. Here, we give the results for the derivatives of the unit vectors,

$$\dot{\boldsymbol{e}}_r = \dot\theta\cos\phi\;\boldsymbol{e}_\theta + \dot\phi\;\boldsymbol{e}_\phi\;, \quad \dot{\boldsymbol{e}}_\theta = -\dot\theta\cos\phi\;\boldsymbol{e}_r + \dot\theta\sin\phi\;\boldsymbol{e}_\phi\;, \quad \dot{\boldsymbol{e}}_\phi = -\dot\phi\;\boldsymbol{e}_r - \dot\theta\sin\phi\;\boldsymbol{e}_\theta\;,$$

and for the kinematic vectors

$$\begin{aligned}
\boldsymbol{r} &= r\,\boldsymbol{e}_r\\
\boldsymbol{v} &= \dot r\boldsymbol{e}_r + r\dot\theta\cos\phi\,\boldsymbol{e}_\theta + r\dot\phi\,\boldsymbol{e}_\phi\\
\boldsymbol{a} &= (\ddot r - r\dot\theta^2\cos^2\phi - r\dot\phi^2)\,\boldsymbol{e}_r\\
&+ (2\dot r\dot\theta\cos\phi + r\ddot\theta\cos\phi - 2r\dot\theta\dot\phi\sin\phi)\,\boldsymbol{e}_\theta\\
&+ (2\dot r\dot\phi + r\dot\phi^2\sin\phi\,\cos\phi + r\ddot\phi)\;\boldsymbol{e}_\phi\;.
\end{aligned}$$

Equations of Motion

Finally, in spherical $r\theta\phi$ coordinates, we write $\boldsymbol{F} = F_r\boldsymbol{e}_r + F_\theta\boldsymbol{e}_\theta + F_\phi\boldsymbol{e}_\phi$ and $\boldsymbol{a} = a_r\boldsymbol{e}_r + a_\theta\boldsymbol{e}_\theta + a_\phi\boldsymbol{e}_\phi$. Thus,

$$\begin{aligned}
F_r &= m\,a_r = m\,(\ddot r - r\dot\theta^2\cos^2\phi - r\dot\phi^2)\\
F_\theta &= m\,a_\theta = m\,(2\dot r\dot\theta\cos\phi + r\ddot\theta\cos\phi - 2r\dot\theta\dot\phi\sin\phi)\\
F_\phi &= m\,a_\phi = m\,(2\dot r\dot\phi + r\dot\phi^2\sin\phi\cos\phi + r\ddot\phi)\;.
\end{aligned}$$

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

2/6, 2/7, 3/5