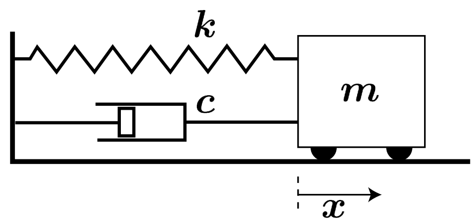
Lecture D32 : Damped Free Vibration

Spring-Dashpot-Mass System



Spring Force $F_s = -kx$, $k > 0$

Dashpot $F_d = -c\dot{x}$, $c > 0$

Newton's Second Law ($m\ddot{x} = \sum F$)

$$m\ddot{x} + c\dot{x} + kx = 0$$

(Define) Natural Frequency $\omega_n = \sqrt{k/m}$, and Period $\tau = 2\pi/\omega_n$

(Define) Damping Factor $\zeta = c/(2m\omega_n)$

Equation of motion

$$\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0}$$

Solution

Try

$$x(t) = Ae^{\lambda t}$$

Characteristic Polynomial

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

Roots

$$\lambda_1 = \omega_n(-\zeta + \sqrt{\zeta^2 - 1}), \quad \lambda_2 = \omega_n(-\zeta - \sqrt{\zeta^2 - 1})$$

General Solution (superposition)

$$\begin{aligned} x &= A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} \\ &= A_1 e^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} + A_2 e^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t} \end{aligned}$$

Types of Solutions

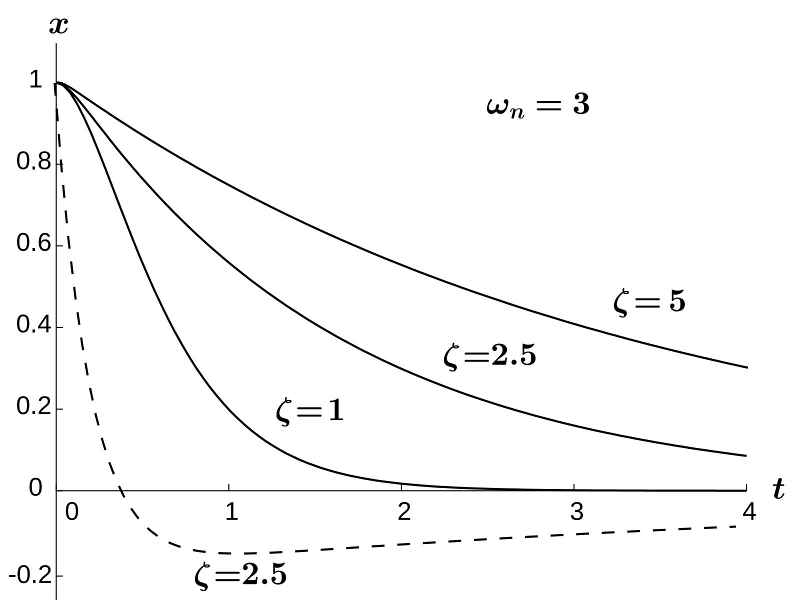
- $\zeta > 1$ (**Overdamped**)

$$\sqrt{\zeta^2 - 1} > 0 \quad \Rightarrow \quad \lambda_{1,2} < 0$$

- $\zeta = 1$ (**Critically Damped**)

$$\lambda_1 = \lambda_2 = -\omega_n < 0$$

$$x(t) = (A_1 + A_2 t)e^{-\omega_n t}$$



One zero crossing at most !!

Types of Solutions (cont'd)

- $\zeta < 1$ (**Underdamped**)

  Damped Natural Frequency (and Period)

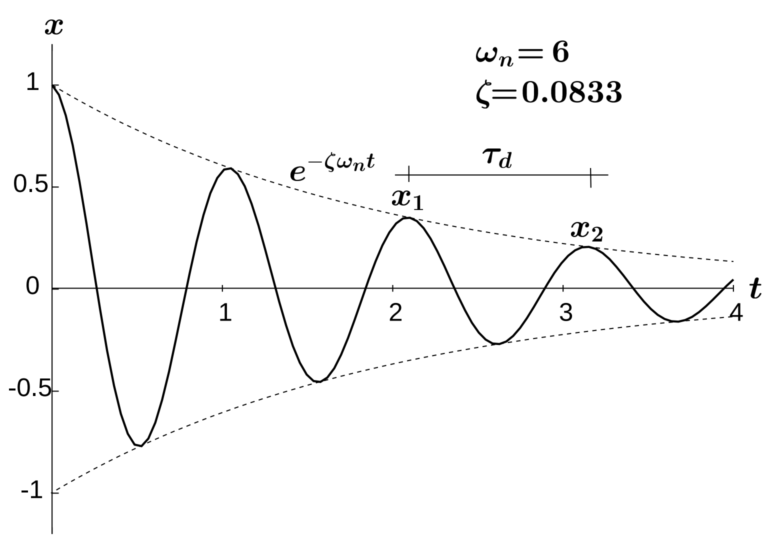
  $$\omega_d = \omega_n \sqrt{1 - \zeta^2}, \quad \tau_d = \frac{2\pi}{\omega_d}$$

  Solution

  $$x(t) = e^{-\zeta \omega_n t}(A \cos \omega_d t + B \sin \omega_d t)$$

  or,

  $$x(t) = C e^{-\zeta \omega_n t} \sin(\omega_d t + \phi)$$



Damping Factor

Needs to be estimated experimentally:

Measure the ratio of two (or more) successive amplitudes x_1 and x_2,

$$\frac{x_1}{x_2} = \frac{Ce^{-\zeta\omega_n t_1}}{Ce^{-\zeta\omega_n (t_1+\tau_d)}} = e^{\zeta\omega_n \tau_d}$$

Let $\delta = \ln(x_1/x_2)$. Since $\omega_d = \omega_n\sqrt{1-\zeta^2}$,

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$$

For lightly damped systems,

$$\zeta \approx \frac{\delta}{2\pi}$$

Energy Decay

For **Undamped** harmonic oscillator:

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_{max}^2 \quad \text{(constant)}$$

For **Underdamped** oscillator

$$x_{max}(t) = x_{max}(0)e^{-\zeta\omega_n t}$$

Thus, we expect

$$\boxed{E(t) = E(0)e^{-2\zeta\omega_n t}}$$

. . . valid for lightly damped systems

This can be used to estimate ζ if Energy (decay) can be measured.