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Lecture D29 - Central Force Motion: Orbits

In lecture D28, we derived three basic relationships embodying Kepler's laws:

- Equation for the orbit trajectory,

$$r = \frac{h^2/\mu}{1+e\cos\theta} \quad \underbrace{\left(= \frac{a(1-e^2)}{1+e\cos\theta} \right)}_{\textit{elliptical orbits}} . \tag{1}$$

- Conservation of angular momentum,

$$h = r^2\dot{\theta} = |\boldsymbol{r} \times \boldsymbol{v}| . \tag{2}$$

- Relationship between the major semi-axis and the period of an elliptical orbit,

$$\mu = \left(\frac{2\pi}{T}\right)^2 a^3. \tag{3}$$

In this lecture, we will first derive an additional useful relationship expressing conservation of energy, and then examine different types of trajectories.

Energy Integral

Since there are no dissipative mechanisms and the only force acting on m can be derived from a gravitational potential, the total energy for the orbit will be conserved. Recall that the gravitational potential per unit mass is given by $-\mu/r$. That is, $\boldsymbol{F}/m = -\boldsymbol{\nabla}(-\mu/r) = -(\mu/r^2)\boldsymbol{e}_r$. Note that the origin for the gravitational potential is taken to be at infinity. Therefore, for finite values of r, the potential is negative. The kinetic energy per unit mass is $v^2/2$. Therefore,

$$\frac{1}{2}v^2 - \frac{\mu}{r} = E \equiv \text{constant} .$$

The total specific energy, E, can be related to the parameters defining the trajectory by evaluating the total energy at the orbit's periapsis ($\theta = 0$). From equation 1, $r_\pi = (h^2/\mu)/(1+e)$, and, from equation 2, $v_\pi^2 = h^2/r_\pi^2 = \mu(1+e)/r_\pi$, since at the periapsis r_π and v_π are orthogonal. Thus,

$$E = \frac{1}{2}v_\pi^2 - \frac{\mu}{r_\pi} = \frac{\mu^2}{2h^2}(e^2-1). \tag{4}$$

We see that the value of the eccentricity determines the sign of E. In particular, for

$e < 1$	the trajectory is closed (ellipse), and	$E < 0$,
$e = 1$	the trajectory is open (parabola), and	$E = 0$,
$e > 1$	the trajectory is open (hyperbola), and	$E > 0$.

Equations 1, 2, and 3, together with the energy integral 4, provide most of relationships necessary to solve basic engineering problems in orbital mechanics.

Types of Orbits

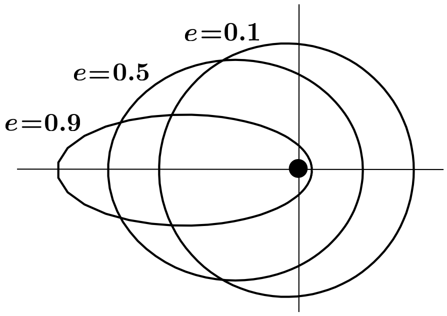
Elliptic Orbits ($e < 1$)

When the trajectory is elliptical, $h^2 = a\mu(1-e^2)$ (see lecture D28). Then, the total specific energy simplifies to $E = -\mu/(2a)$, and the conservation of energy can be expressed as

$$\frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}. \tag{5}$$

This expression shows that the energy (and the period) of an elliptical orbit depends only on the major semi-axis. We also see that for a fixed a, the value of h determines the eccentricity. There are two limiting cases: $e \to 1$, which gives $h \to 0$, which in turn implies that the minor semi-axis of the ellipse $b \to 0$; and $e = 0$ which corresponds to a circular orbit with $h = \sqrt{a\mu}$. In the first case, the maximum value of the eccentricity is limited by the size of the planet, since, for sufficiently large values of e, the trajectory will collapse onto the planet's surface.

Below we show three elliptical trajectories that have the same energy (same value of a), but different eccentricities.



Circular Orbits ($e = 0$)

This is a particular case of an elliptic orbit. The energy equation is given by equation 5. The radius is constant

$$r = \frac{h^2}{\mu} = \frac{v_c^2 r^2}{\mu}.$$

For orbits around the earth, $\mu = gR^2$, where g is the acceleration of gravity at the earth's surface, and R is the radius of the earth. Then,

$$v_c^2 = \frac{\mu}{r} = \frac{gR^2}{r}, \tag{6}$$

which shows that the velocity of a circular orbit is inversely proportional to the radius. We now consider two particular orbits of interest:

1) $\underline{r = R}$

   This corresponds to a hypothetical satellite orbiting the earth at a zero altitude above the earth's surface. The orbit's velocity is

$$v_c = \sqrt{gR} = 7910\,\mathrm{m/s},$$

   and the period, from equation 3, is

$$T = \frac{2\pi}{\sqrt{gR^2}} R^{3/2} = 2\pi\sqrt{\frac{R}{g}} = 84.4\,\mathrm{min}\ .$$

   This period is called the "Schuler" period, and it is the minimum period that any free flight object can have in orbit around the earth.

2) Synchronous Orbits

   These are orbits whose period is the same as the earth's rotational period (24 h). In addition, if the orbit is in the equatorial plane, the orbit is said to be *geostationary* because the satellite will stay fixed relative to an observer on the earth. Using equation 3,

$$a = \left(\frac{T^2 g R^2}{4\pi^2}\right)^{1/3} = 42042\,\mathrm{km} \approx 6.6R,$$

   which corresponds to an altitude above the surface of $5.6R$.

Example ***Elliptical Orbits***

Consider a satellite launched from an altitude d above the earth's surface, with velocity $v_c = \sqrt{\mu/(R+d)}$. If the direction of the velocity is orthogonal to the position vector, the trajectory will clearly be a circular orbit of radius $R+d$. However, if the velocity is in any other direction, the trajectory will be an ellipse of semi-major axis equal to $R+d$. The characteristics of the ellipse can easily be determined as follows: knowing $\boldsymbol{r}$ and $\boldsymbol{v}$, we can determine h; using equation 4, we can determine e; and from the trajectory equation 1, we can determine θ, and hence the orientation of the ellipse.

Parabolic Orbits $(e = 1)$

From equation 1, we see that $r \to \infty$ for $\theta \to \pi$. From the energy integral, with $E = 0$, we have that,

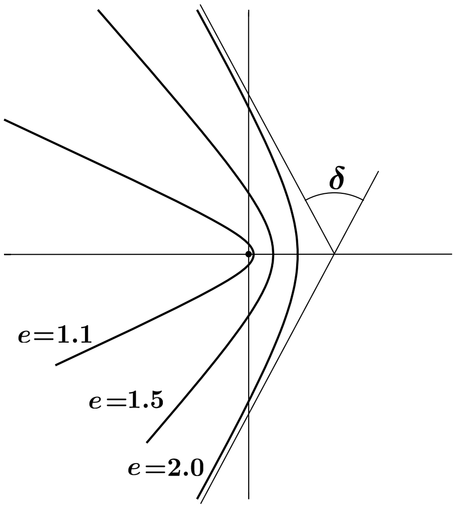
$$\frac{1}{2}v_e^2 - \frac{\mu}{r} = 0, \qquad v_e^2 = \frac{2\mu}{r}. \tag{7}$$

Here, v_e is the escape velocity and is the smallest velocity needed to escape the field of gravitational attraction. Comparing equations 6 and 7, we see that, for a given r, the escape velocity is a factor of $\sqrt{2}$ larger than the velocity necessary to maintain a circular orbit. Thus, if a satellite is on a circular orbit with velocity v_c, the necessary Δv to escape is $(\sqrt{2}-1)v_c$.

It should be noted that a satellite in a parabolic trajectory has a total specific energy, E, equal to zero. This means that when r increases, the kinetic energy is transformed to potential energy such that, at infinity, the residual velocity is equal to zero.

Hyperbolic Trajectory $(e > 1)$

For $\theta_\infty \to \pm\cos^{-1}(1/e)$, we have $r \to \infty$. Hence, the trajectories are open. Moreover, if the velocity v, at a given r is known, the residual velocity is simply $v_\infty = v - v_e = v - \sqrt{2\mu/r} = \sqrt{2E}$. For a given energy level, the eccentricity of the orbit is determined by h. An important parameter for hyperbolic orbits is the turning angle, δ, which is the angle through which the velocity changes along the trajectory as the body travels from $-\infty$ to ∞. The turning angle is given by $\delta = 2(\theta_\infty - \pi)$, or $\delta = 2\sin^{-1}(1/e)$. Below we show several hyperbolic trajectories which have identical terminal velocities for different values of the eccentricity (and turning angle).

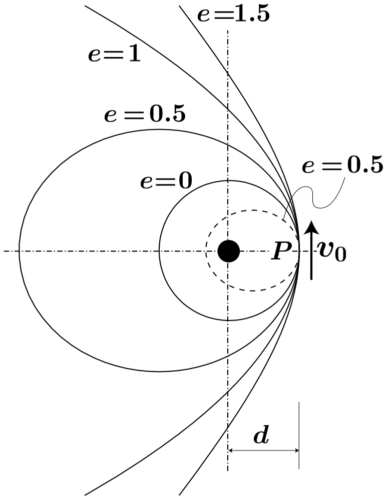


Example ***Different orbits as a function of*** v_0

We consider the problem of launching a satellite at an altitude d with an initial velocity v_0, along the direction tangent to the earth's surface. We consider the different trajectories that are obtained as we vary the magnitude of v_0.

For $v_0 = v_{0c} \equiv \sqrt{\mu/(R+d)}$, the trajectory will be a circle $(e = 0)$. For $v_0 = v_{0e} \equiv \sqrt{2\mu/(R+d)}$, the trajectory will be parabola $(e = 1)$. For $v_0 > v_{0e}$, the trajectory will be a hyperbola, whereas for

$v_{0c} < v_0 < v_{0e}$ the trajectory will be elliptical. We note that, for all these orbits, the launch point, P, is the orbit's perigee, or the closest point in the trajectory to the earth's center.



On the other hand, when the velocity $v_0 < v_{0c}$, the straightforward use of expressions 1 and 2 gives a negative eccentricity! The eccentricity is negative because equation 1 assumes that the origin of θ is taken to be at the orbit's perigee. In turns out that in this case, the orbit has a lower energy than the circular orbit, and, hence, the launch point is now the orbit's apogee. The proper use of equation 1 requires that $\theta = \pi$. In this case, we have $d + R = (v_0^2(d+R)^2/\mu)/(1 + e\cos\pi)$, which for $v_0 < v_{0c}$ gives a positive eccentricity. In the picture above, we see one such trajectory depicted with a dotted line.

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

3/13 (energy analysis)